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Walks on graphs and their connections with tensor invariants and centralizer algebras

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ABSTRACT

The number of walks of k steps from the node $\mathsf{0}$ to the node λ on the McKay quiver determined by a finite group G and a G-module V is the multiplicity of the irreducible G-module G_λ in the tensor power $\mathsf{V}^{\otimes k}$, and it is also the dimension of the irreducible module labeled by λ for the centralizer algebra $\mathsf{Z}_k(\mathsf{G}) = \mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes k})$. This paper explores ways to effectively calculate that number using the character theory of G. We determine the corresponding Poincaré series. The special case $\lambda = \mathsf{0}$ gives the Poincaré series for the tensor invariants $\mathsf{T}(\mathsf{V})^{\mathsf{G}} = \bigoplus_{k=0}^{\infty} (\mathsf{V}^{\otimes k})^{\mathsf{G}}$ and a tensor analog of Molien's formula for polynomial invariants. When G is abelian, we show that the exponential generating function for the number of walks is a product of generalized hyperbolic functions. Many graphs (such as circulant graphs) can be viewed as McKay quivers, and the methods presented here provide efficient ways to compute the number of walks on them.

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1. Introduction

Let G be a finite group, and assume that the elements λ of $\Lambda(\mathsf{G})$ index the irreducible complex representations of G, hence also the conjugacy classes of G. Let G_λ denote the irreducible G-module indexed by λ, and let χ_λ be its character. The module G_0 denotes the trivial one-dimensional G-module with $\chi_0(g) = 1$ for all $g \in \mathsf{G}$.

The *McKay quiver* $\mathsf{Q}_\mathsf{V}(\mathsf{G})$ (also known as the *representation graph*) associated to a finite-dimensional G-module V over the complex field $\mathbb{C}$ has nodes corresponding to the irreducible G-modules $\{\mathsf{G}_\lambda \mid \lambda \in \Lambda(\mathsf{G})\}$. For $\nu \in \Lambda(\mathsf{G})$, there are $a_{\nu,\lambda}$ arrows from ν to λ in $\mathsf{Q}_\mathsf{V}(\mathsf{G})$ if

$$\mathsf{G}_\nu \otimes \mathsf{V} = \bigoplus_{\lambda \in \Lambda(\mathsf{G})} a_{\nu,\lambda} \mathsf{G}_\lambda. \tag{1}$$

If $a_{\nu,\lambda} = a_{\lambda,\nu}$, then we draw $a_{\nu,\lambda}$ edges without arrows between ν and λ. The number of arrows $a_{\nu,\lambda}$ from ν to λ in $\mathsf{Q}_\mathsf{V}(\mathsf{G})$ is the multiplicity of G_λ as a summand of $\mathsf{G}_\nu \otimes \mathsf{V}$. Since each step on the graph is achieved by tensoring with V,

$$\begin{aligned} \mathsf{m}_k^\lambda &:= \text{number of walks of } k \text{ steps from } \mathsf{0} \text{ to } \lambda \\ &= \text{multiplicity of } \mathsf{G}_\lambda \text{ in } \mathsf{G}_0 \otimes \mathsf{V}^{\otimes k} \cong \mathsf{V}^{\otimes k}. \end{aligned}$$

For a faithful G-module V, any irreducible G-module G_λ occurs in $\mathsf{V}^{\otimes \ell}$ for some ℓ by Burnside's theorem (in fact, for some ℓ such that $0 \le \ell \le |\mathsf{G}|$ by Brauer's strengthening of that result [8, Thm. 9.34]). This implies that there is a directed path with ℓ steps from G_0 to G_λ in $\mathsf{Q}_\mathsf{V}(\mathsf{G})$.

The *centralizer algebra*,

$$\mathsf{Z}_k(\mathsf{G}) = \{z \in \mathsf{End}(\mathsf{V}^{\otimes k}) \mid z(g.w) = g.z(w) \ \ \forall\, g \in \mathsf{G}, w \in \mathsf{V}^{\otimes k}\},$$

plays a critical role in studying $\mathsf{V}^{\otimes k}$, as it contains the projection maps onto the irreducible summands of $\mathsf{V}^{\otimes k}$.

Let $\Lambda_k(\mathsf{G})$ denote the subset of $\Lambda(\mathsf{G})$ corresponding to the irreducible G-modules that occur in $\mathsf{V}^{\otimes k}$ with multiplicity at least one. *Schur–Weyl duality* establishes essential connections between the representation theories of G and $\mathsf{Z}_k(\mathsf{G})$:

- $\mathsf{Z}_k(\mathsf{G})$ is a semisimple associative $\mathbb{C}$-algebra whose irreducible modules $\mathsf{Z}_k^\lambda(\mathsf{G})$ are in bijection with the elements λ of $\Lambda_k(\mathsf{G})$.
- $\dim \mathsf{Z}_k^\lambda(\mathsf{G}) = \mathsf{m}_k^\lambda$, the number of walks of k steps from the trivial G-module G_0 to G_λ on $\mathsf{Q}_\mathsf{V}(\mathsf{G})$.
- If $d_\lambda = \dim \mathsf{G}_\lambda$, then the tensor space $\mathsf{V}^{\otimes k}$ has the following decompositions:

$$\mathsf{V}^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda_k(\mathsf{G})} \mathsf{m}_k^\lambda\, \mathsf{G}_\lambda \qquad \text{as a } \mathsf{G}\text{-module},$$