Journal of Algebra 509 (2018) 101–104

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On the torsion rank of divisible multiplicative groups of fields

Ethan Robinett

Department of Mathematics, University of California, Santa Barbara, CA 93106, United States

A R T I C L E I N F O

Article history:
Received 7 September 2017
Available online 18 May 2018
Communicated by Kazuhiko Kurano

Keywords:
p-Divisible systems
Divisible abelian groups
Multiplicative groups of fields
Torsion-free rank
Torsion rank

A B S T R A C T

In his characterization of the divisible abelian groups which are isomorphic to the unit group of a field, Greg Oman asked the following question: if G is a divisible abelian group that can be realized as the multiplicative group of a field, must the torsion rank of G be either 0 or infinite? The purpose of this paper is to provide a positive answer to this question.

1. Introduction

In [7], Greg Oman partially answers a decades-old question initially posed by Laszlo Fuchs in [4], namely: which abelian groups are isomorphic to the unit group of a field? Oman tackles this problem for the class of divisible abelian groups, extending previous results of Adler [1] and Contessa, Mott and Nichols [2] (for further information regarding the general problem posed by Fuchs and related results, the reader is referred to [3], [5], [6], and [8]). Of particular importance in the argument is a lemma contained in [2] (also proven in [7]).

E-mail address: robinett@math.ucsb.edu.

https://doi.org/10.1016/j.jalgebra.2018.04.035

Proposition 1 *(Lemma 3 in [7] and Corollary 2.4 in [2]). Let G be an abelian group with finite, nonzero torsion-free rank. Then G is not isomorphic to the multiplicative group of any field.*

Here, the *torsion-free rank* of G is the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} G$. In the interest of resolving any potential ambiguity, we briefly define the related notion of the *torsion rank* of an abelian group G. Following Fuchs in [4], we say a subset A of G is linearly independent if A does not contain 0 and whenever we have $n_1 a_1 + \cdots + n_k a_k = 0$ for $n_i \in \mathbb{Z}$ and $a_i \in A$, then $n_i a_i = 0$ for each i. The *rank* of G is then defined to be the cardinality of a maximal linearly independent subset consisting only of elements of either infinite or prime-power order. As one may expect, we then define the *torsion rank* of G to be the rank of $T(G)$, the torsion subgroup of G.

If G is additionally assumed to be divisible, then we have $G \cong T(G) \oplus G/T(G)$ and Proposition 1 implies that if the summand $G/T(G)$ has finite, nonzero rank, then G is not realizable as the multiplicative group of a field. It is this statement which naturally prompts a question that is similar in spirit.

Question 1. *Let G be a divisible abelian group that is isomorphic to the multiplicative group of a field. Is the torsion rank of G either 0 or infinite?*

It is further remarked in [7] that the existence of infinitely many Fermat primes, a currently wide open problem, implies an affirmative answer to this question in characteristics other than 2. Oman conjectures a positive answer in general and this is what we endeavor to prove here.

2. Proof of the result

An affirmative answer to Question 1 follows by quite elementary means once we collect a few definitions and facts. In what follows, $\mathbb{Z}^+$ denotes the set of strictly positive integers and $\mathbb{N}$ the set of non-negative integers. We first recall from [7] the notion of a *p-divisible system*.

Definition. Let $S \subseteq \mathbb{Z}^+$ be nonempty and let p be a prime. Then S is a *p-divisible system* if and only if

1. For each $a \in S$ and $x > 0$: if x divides a, then $x \in S$.
2. If $a, b \in S$, then $\operatorname{lcm}(a, b) \in S$.
3. If $a \in S$ and q is a prime such that $p^a \equiv 1 \pmod{q}$, then $q^n \in S$ for every $n \in \mathbb{Z}^+$.

These *p-divisible systems* are intimately related to the question at hand, as the following statement shows.