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Varieties of elements of given order in simple algebraic groups

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A R T I C L E I N F O

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A B S T R A C T

Given a positive integer u and a simple algebraic group G defined over an algebraically closed field K of characteristic p, we derive properties about the subvariety $G_{[u]}$ of G consisting of elements of G of order dividing u. In particular, we determine the dimension of $G_{[u]}$, completing results of Lawther [7] in the special case where G is of adjoint type. We also apply our results to the study of finite simple quotients of triangle groups, giving further insight on a conjecture we proposed in [10] as well as proving that some finite quasisimple groups are not quotients of certain triangle groups.

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1. Introduction

Let G be a reductive algebraic group defined over an algebraically closed field K of characteristic p (possibly equal to 0), C be a conjugacy class of G and u be a positive integer. In 2007 Guralnick [3] proved the following result:

Theorem 1 *([3, Theorem 1.1]). Given a reductive algebraic group G over an algebraically closed field, a conjugacy class C of G and a positive integer u, the set $\{g \in G : g^u \in C\}$ is a finite union of conjugacy classes of G.*

In this paper, we concentrate our attention to the case where G is connected and $C = \{1\}$ is the trivial conjugacy class of G. For a positive integer u, we let

$$G_{[u]} = \{g \in G : g^u = 1\}$$

be the subvariety of G consisting of elements of G of order dividing u and set $j_u(G) = \dim G_{[u]}$. We also let $d_u(G)$ be the minimal dimension of a centralizer in G of an element of G of order dividing u. We are merely interested in determining $j_u(G)$ for every positive integer u (when G is a simple algebraic group).

For completeness, we begin by proving Guralnick's result in the case where G is connected and $C = \{1\}$. In the statement below, given $g \in G$, we let g^G denote the conjugacy class of g in G.

Proposition 2. *Let G be a connected reductive algebraic group defined over an algebraically closed field K of characteristic p. Let u be a positive integer. Then the number of conjugacy classes of G of elements of order dividing u is finite. In particular $G_{[u]}$ is a finite union of conjugacy classes of G. Moreover $\dim G_{[u]} = \max_{g\in G_{[u]}} \dim g^G$ and* $\mathrm{codim}\ G_{[u]} = d_u(G)$.

Given a simple algebraic group G defined over an algebraically closed field K of characteristic p, we denote by $G_{s.c.}$ (respectively, $G_{a.}$) the simple algebraic group over K of simply connected (respectively, adjoint) type having the same Lie type and Lie rank as G. We prove the following result partially proved in [7, Theorem 3.11]: