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Counting the number of distinct distances of elements in valued field extensions

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ABSTRACT

The defect of valued field extensions is a major obstacle in open problems in resolution of singularities and in the model theory of valued fields, whenever positive characteristic is involved. We continue the detailed study of defect extensions through the tool of distances, which measure how well an element in an immediate extension can be approximated by elements from the base field. We show that in several situations the number of essentially distinct distances in fixed extensions, or even just over a fixed base field, is finite, and we compute upper bounds. We apply this to the special case of valued functions fields over perfect base fields. In particular, this provides important information used in forthcoming research on the ramification theory of two-dimensional valued function fields.

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1. Introduction

By $(L|K, v)$ we denote a field extension $L|K$ where v is a valuation on L and K is endowed with the restriction of v. The valuation ring of v on L will be denoted by $\mathcal{O}_L$, and that on K by $\mathcal{O}_K$. The value group of (L, v) will be denoted by vL, and its residue field by Lv. The value of an element a will be denoted by va, and its residue by av.

The **defect**, also known as **ramification deficiency**, of finite extensions $(L|K, v)$ of valued fields is a phenomenon that only appears when the residue field Kv has positive characteristic. It is a main obstacle to the solution of deep open problems in positive characteristic, such as:

• local uniformization (the local form of resolution of singularities), which is not known for arbitrary dimension in positive characteristic,

• the model theory of valued fields, in particular the open question whether Laurent series fields over finite fields have a decidable theory.

Both problems are linked through the structure theory of valued function fields, in which it is essential to tame the defect, as well as wild ramification, cf. [9,12,14–16]. While implicitly known through the work of algebraic geometers and model theorists since the 1950s, the connection of the defect with the problem of local uniformization and the model theory of valued fields with positive residue characteristic has been pointed out in detail in the cited works of the second author. Defects also appear in crucial examples, as in the paper [4].

Using tools of ramification theory, the study of extensions of valued fields of residue characteristic $p > 0$ with nontrivial defect can be reduced to the study of normal extensions of degree p with nontrivial defect. Such extensions are immediate. An arbitrary extension $(L|K, v)$ of valued fields is **immediate** if the canonical embeddings of vK in vL and of Kv in Lv are onto. As a consequence, for every $a \in L \setminus K$ the set

$$v(a - K) \;:=\; \{v(a - c) \mid c \in K\}$$

does not have a maximal element; this follows from [8, Theorem 1]. If a is an element of any valued field extension of (K, v) such that $v(a - K)$ has no maximal element, then this set is an initial segment of vK. We associate with it a cut in the divisible hull $\widetilde{vK}$ of vK by taking as the lower cut set the smallest initial segment in $\widetilde{vK}$ which contains $v(a - K)$. This cut is called the **distance of** a **over** K and denoted by $\operatorname{dist}(a, K)$. For more details, see Section 2.2.

Distances can be used to classify defect extensions. If an extension $L|K$ of degree p is Galois and the field K is itself of characteristic p, then $L|K$ is an **Artin–Schreier extension**, that is, L is generated over K by an element ϑ such that

$$\vartheta^p - \vartheta \;\in\; K\,; \tag{1}$$