Journal of Algebra 509 (2018) 212–239

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

# Interpolating between Hilbert–Samuel and Hilbert–Kunz multiplicity

William D. Taylor

*Department of Mathematical Sciences, University of Arkansas, Fayetteville, AR 72701, United States of America*

ARTICLE INFO

*Article history:*
Received 16 August 2017
Available online 19 May 2018
Communicated by Steven Dale Cutkosky

*Keywords:*
Commutative algebra
Hilbert–Kunz multiplicity
Hilbert–Samuel multiplicity
Interpolation
Positive characteristic

ABSTRACT

We define a function, called $s$-multiplicity, that interpolates between Hilbert–Samuel multiplicity and Hilbert–Kunz multiplicity by comparing powers of ideals to the Frobenius powers of ideals. The function is continuous in $s$, and its value is equal to Hilbert–Samuel multiplicity for small values of $s$ and is equal to Hilbert–Kunz multiplicity for large values of $s$. We prove that it has an Associativity Formula generalizing the Associativity Formulas for Hilbert–Samuel and Hilbert–Kunz multiplicity. We also define a family of closures such that if two ideals have the same $s$-closure then they have the same $s$-multiplicity, and the converse holds under mild conditions. We describe the $s$-multiplicity of monomial ideals in toric rings as a certain volume in real space.

© 2018 Elsevier Inc. All rights reserved.

## 1. Introduction

The purpose of this paper is to investigate a function that interpolates continuously between Hilbert–Samuel multiplicity and Hilbert–Kunz multiplicity. First we define a limit that behaves like a multiplicity, then we normalize it to get a proper interpolation

*E-mail address:* wdtaylor@uark.edu.

https://doi.org/10.1016/j.jalgebra.2018.05.015
0021-8693/© 2018 Elsevier Inc. All rights reserved.

between the Hilbert–Samuel and Hilbert–Kunz multiplicities. This interpolating function, which we call $s$-multiplicity, is a single object which captures the behavior of both multiplicities as well as a family of multiplicity-like functions between them. Many of the similarities between the two multiplicities, such as the existence of an Associativity Formula and the connection to a closure, can be interpreted as special cases of a more general statement about $s$-multiplicity.

Throughout the paper we will assume that all rings are noetherian and equicharacteristic. By $\lambda_R(M)$ we mean the length of $M$ as an $R$-module. When the ring $R$ is understood we may write $\lambda(M)$ for $\lambda_R(M)$.

**Definition 1.1.** Let $(R, \mathfrak{m})$ be a local ring of dimension $d$, $I \subseteq R$ an $\mathfrak{m}$-primary ideal of $R$, and $M$ a finitely generated $R$-module. The *Hilbert–Samuel multiplicity of $M$ with respect to $I$* is defined to be

$$e(I; M) = \lim_{n\to\infty} \frac{d! \cdot \lambda(M/I^n M)}{n^d}.$$

We often write $e(I)$ for $e(I; R)$.

Many properties of the Hilbert–Samuel multiplicity are well known. For instance, if $I \subseteq J$ are ideals that have the same integral closure, then $e(I) = e(J)$, and if $R$ is formally equidimensional, then the converse holds [9]. The Hilbert–Samuel multiplicity is always a nonnegative integer, $e(\mathfrak{m}) = 1$ if $(R, \mathfrak{m})$ is regular, and if $R$ is formally equidimensional the converse holds [7, Theorem 40.6].

When $R$ is of prime characteristic $p > 0$, the Frobenius map $F : R \to R$ taking $r \mapsto r^p$ is a ring homomorphism, and so we may treat $R$ as a module over itself via the action $r \cdot x = r^p x$. In this case, we often denote the module $R$ with this new action by $F_*R$, and elements of this module by $F_*r$ for $r \in R$. An $R$-module homomorphism $\varphi : F_*R \to R$ is called a $p^{-1}$-linear map, and has the property that for any $r, x \in R$, $r\varphi(F_*x) = \varphi(F_*(r^p x))$. If $F_*R$ is finitely generated as an $R$-module, we say the ring $R$ is $F$-finite. For an ideal $I \subseteq R$ and $e \in \mathbb{N}$, the $e$th Frobenius power of $I$, denoted $I^{[p^e]}$, is the ideal generated by the $p^e$-th powers of the elements of $I$, equivalently by the $p^e$-th powers of a set of generators for $I$. For any $p^{-1}$-linear map $\varphi$ and ideal $I \subseteq R$, $\varphi(F_*(I^{[p]})) \subseteq I$.

When $R$ is a ring of positive characteristic, we can define a limit similar to the Hilbert–Samuel multiplicity using the Frobenius powers of the ideal instead of the powers.

**Definition 1.2.** Let $(R, \mathfrak{m})$ be a local ring of dimension $d$, $I \subseteq R$ an $\mathfrak{m}$-primary ideal of $R$, and $M$ a finitely generated $R$-module. The *Hilbert–Kunz multiplicity of $M$ with respect to $I$* is defined to be

$$e_{HK}(I; M) = \lim_{e\to\infty} \frac{\lambda\big(M/I^{[p^e]}M\big)}{p^{ed}}.$$

We often write $e_{HK}(I)$ for $e_{HK}(I; R)$.

Download English Version:

https://daneshyari.com/en/article/8895887

Download Persian Version:

https://daneshyari.com/article/8895887

Daneshyari.com