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# Class 2 quotients of solvable linear groups

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ABSTRACT

Let $G$ be a finite group, and let $V$ be a completely reducible faithful $G$-module. By a result of Glauberman it has been known for a long time that if $G$ is nilpotent of class 2, then $|G| < |V|$. In this paper we generalize this result as follows. Assuming $G$ to be solvable, we show that the order of the maximal class 2 quotient of $G$ is strictly bounded above by $|V|$.



## 1. Introduction

In [3, Proposition 1] G. Glauberman proved that if $n$ is a positive integer, $p$ is a prime and $G \leq \mathrm{GL}(n, p)$ is a $p'$-group which is nilpotent of class 2, then $|G| < |V|$.

The goal of this paper is to generalize this result as follows. For a finite group $G$ put $G^c = [G, G, G] = [G', G]$; i.e., $G^c$ is the intersection of all normal subgroups of $G$ whose quotient group is nilpotent of class 2, so that $G/G^c$ is the (maximal) class 2 quotient of $G$. With this we will prove the following generalization of Glauberman's result.

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**1.1. Theorem.** *Let $G$ be a finite solvable group and $V \neq 0$ a finite faithful completely reducible $G$-module, possibly of mixed characteristic. Then*

$$|G/G^c| \ < \ |V|.$$

Of course, when $G$ is nilpotent of class 2, this is just Glauberman's result, which we will use in the proof of the above theorem.

Theorem 1.1 can also be viewed as a strengthening – for solvable groups and completely reducible modules – of a result by Aschbacher and Guralnick, see [2, Theorem 1]. They proved that the order of the abelian quotient $|G/G'|$ of $G$, i.e., the class 1 quotient of $G$, is strictly bounded above by $|V|$, where $G$ is a finite faithful linear group on the finite module $V$ such that $\mathrm{O}_r(G) = 1$ for the characteristic $r$ of $V$. For solvable $G$ and completely reducible $V$ our new result shows that even the class 2 quotient of $G$ is strictly bounded above by $|V|$.

We note that we believe that the main result of this paper remains true for arbitrary finite groups in place of solvable groups.

## 2. Bounding the class 2 quotient by the module size

In this section we prove Theorem 2.3. We first prove a reduction lemma.

**2.1. Lemma.** *Let $G$ be a finite group and $N \trianglelefteq G$. Then*

$$|G/G^c| = |G/G^cN| \cdot |N : N \cap G^c|;$$

*and*

$$|G : G^c| \text{ divides } |G/N : (G/N)^c||N : N^c|.$$

**Proof.** The proof goes along exactly the same lines as the proof of [7, Lemma 1]. □

In [7] the following result is proved, albeit not stated as a separate result.

**2.2. Lemma.** *Let $G$ be a finite solvable group and $V$ a finite faithful completely reducible $G$-module, possibly of mixed characteristic. If $G$ is not nilpotent, then*

$$|G/G'| \ < \ |F(G)/F(G)'|.$$

**Proof.** For the convenience of the reader we present a self-contained proof of this result here, even though the main ideas of the proof are contained in the proof of [7, Theorem 2.3]. So suppose $G$ is not nilpotent. Write $F = F(G)$ for the Fitting subgroup and $\Phi = \Phi(G)$ for the Frattini subgroup of $G$.