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Deforming spaces of m-jets of hypersurfaces singularities

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ABSTRACT

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, and V a hypersurface defined by an irreducible polynomial f with coefficients in $\mathbb{K}$.
In this article we prove that an Embedded Deformation of V which admits a Simultaneous Embedded Resolution induces, under certain conditions, a deformation of the space of m-jets V_m, $m \geq 0$. An example of an Embedded Deformation of V which admits a Simultaneous Embedded Resolution is a $\Gamma(\mathrm{f})$-deformation of V, where V has at most one isolated singularity, and f is non-degenerate with respect to the Newton Boundary $\Gamma(\mathrm{f})$.

1. Introduction

Let V be an algebraic variety over a field $\mathbb{K}$, and m a positive integer. Intuitively the Space of m-jets, V_m, (resp. Space of arcs, V_∞) is the set of morphisms

$$\operatorname{Spec} \mathbb{K}[t]/(t^{m+1}) \to V \ (\text{resp. } \operatorname{Spec} \mathbb{K}[[t]] \to V)$$

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equipped with a "natural" structure of $\mathbb{K}$-schema. During the late 60s Nash studied these spaces as a tool to understand the local geometry of the singular locus of V. He was specifically interested in understanding the *Essential Divisors over V* (see Section 2.2.2). He constructed an injective application, known as the *Nash Application*, from the set of *Nash Components of V* (see Section 2.2.1) to the set of Essential Divisors. Then the question raised was: Is the Nash application bijective?

Obtaining a complete answer to this question took many decades. In the year 2003 Ishii and Kollar (see [16]) showed the first example of a variety such that its Nash application is not surjective. It is worth mentioning that this example is a hypersurface of dimension 4. In the year 2012 Fernandez de Bobadilla and Pe-Pereira (see [10]) gave an affirmative answer to the question for the case of surfaces over $\mathbb{C}$. Finally between the years 2012 and 2013 Johnson, Kollar, and de Fernex constructed examples of hypersurfaces of dimension 3 where the Nash application is not surjective (see [3], [19] and [20]). It is important to mention that many mathematicians worked hard on this problem, making valuable progress with respect to the problem. For example: [4], [9], [12], [14], [16], [17], [18], [23], [24], [26], [27], [30], [36], [37], [38], [39], [40], [41], etc. Unfortunately it is not possible to comment on and cite all the existing works.

However, despite all the progress made, giving an exact description of the image of the Nash application remains an open problem.

Informally speaking, hidden within the Spaces of m-jets and the Space of arcs lies much information on geometry of the subjacent variety. For example:

In the year 1995, Kontsevich, using these spaces, introduced the *motivic integration* to resolve the Batyrev conjecture on the Calabi–Yau varieties (see [21]). For more references on motivic integration see [1], [5], [6] and [28].

Other examples are found in the articles [8], [32] written by Ein and Mustaţă where, amongst other things, it is demonstrated that if V is locally a complete intersection, then V only has rational singularities (resp. log-canonical singularities) if and only if V_m is irreducible (resp. equidimensional) for all $m \geq 0$.

Another interesting application for the Spaces of m-jets and the Space of arcs is obtaining identities and invariants associated to V, for example see the articles [2], and [5].

One of the main subjects of [24] was the study of the following problem: When a "deformation" of the variety V induces a "natural deformation" of the Spaces of m-jets V_m (for greater precision see Section 3.1). In general a deformation of V does not induce a deformation of the spaces of V_m, see Example 1, however there exist important families of examples in which this property is satisfied. For example if V is locally a complete intersection, and only has an isolated singularity of log-canonical type (see Proposition 3.1).

The idea to study the Spaces of m-jets in families is very natural, for example Mourtada in [31] studies families of complex plane branches with constant topological type, and obtains formulas for the calculation of the number and dimension of the irreducible components of the Spaces of m-jets.