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Associated points and integral closure of modules

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A B S T R A C T

Let $X := \mathrm{Spec}(R)$ be an affine Noetherian scheme, and $\mathcal{M} \subset \mathcal{N}$ be a pair of finitely generated R-modules. Denote their Rees algebras by $\mathcal{R}(\mathcal{M})$ and $\mathcal{R}(\mathcal{N})$. Let $\mathcal{N}^n$ be the nth homogeneous component of $\mathcal{R}(\mathcal{N})$ and let $\mathcal{M}^n$ be the image of the nth homogeneous component of $\mathcal{R}(\mathcal{M})$ in $\mathcal{N}^n$. Denote by $\overline{\mathcal{M}^n}$ be the integral closure of $\mathcal{M}^n$ in $\mathcal{N}^n$. We prove that $\mathrm{Ass}_X(\mathcal{N}^n/\overline{\mathcal{M}^n})$ and $\mathrm{Ass}_X(\mathcal{N}^n/\mathcal{M}^n)$ are asymptotically stable, generalizing known results for the case where $\mathcal{M}$ is an ideal or where $\mathcal{N}$ is a free module. Suppose either that $\mathcal{M}$ and $\mathcal{N}$ are free at the generic point of each irreducible component of X or $\mathcal{N}$ is contained in a free R-module. When X is universally catenary, we prove a generalization of a classical result due to McAdam and obtain a geometric classification of the points appearing in $\mathrm{Ass}_X(\mathcal{N}^n/\overline{\mathcal{M}^n})$. Notably, we show that if $x \in \mathrm{Ass}_X(\mathcal{N}^n/\overline{\mathcal{M}^n})$ for some n, then x is the generic point of a codimension-one component of the nonfree locus of $\mathcal{N}/\mathcal{M}$ or x is a generic point of an irreducible set in X where the fiber dimension $\mathrm{Proj}(\mathcal{R}(\mathcal{M})) \to X$ jumps. We prove a converse to this result without requiring X to be universally catenary. Our approach is geometric in spirit. Also, we recover, strengthen, and prove a sort of converse of an important

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result of Kleiman and Thorup about integral dependence of modules.

Contents

1. Introduction

Let $X := \mathrm{Spec}(R)$ be an affine Noetherian scheme and let $\mathcal{M}$ be a finitely generated R-module. Following Eisenbud, Huneke and Ulrich [7], define the *Rees algebra* of $\mathcal{M}$ as the quotient

$$\mathcal{R}(\mathcal{M}) := \mathrm{Sym}(\mathcal{M})/(\cap \mathcal{L}_g)$$

where the intersection is taken over all homomorphisms g from $\mathcal{M}$ to a free R-module $\mathcal{F}_g$ and $\mathcal{L}_g$ denotes the kernel of the induced map $\mathrm{Sym}(\mathcal{M}) \to \mathrm{Sym}(\mathcal{F}_g)$. In fact as shown in Prop. 1.3 [7] $\mathcal{R}(\mathcal{M})$ can be computed from any homomorphism of $\mathcal{M}$ to a free R-module whose dual is surjective.

Let $\mathcal{M} \subset \mathcal{N}$ be a pair of finitely generated R-modules. Assume that $\mathcal{M}$ and $\mathcal{N}$ are either free at the generic point of each irreducible component of X, or they are contained in a free R-module. The inclusion of $\mathcal{M}$ into $\mathcal{N}$ induces a map from $\mathcal{R}(\mathcal{M})$ to $\mathcal{R}(\mathcal{N})$. Denote by $\mathcal{N}^n$ the nth homogeneous component of $\mathcal{R}(\mathcal{N})$ and by $\mathcal{M}^n$ the image of the nth homogeneous component of $\mathcal{R}(\mathcal{M})$ in $\mathcal{R}(\mathcal{N})$. Finally, recall that the *integral closure* $\overline{\mathcal{M}^n}$ of $\mathcal{M}^n$ in $\mathcal{N}^n$ is the module generated by those elements from $\mathcal{N}^n$ that satisfy an equation in $\mathcal{R}(\mathcal{N})$ of integral dependence over $\mathcal{R}(\mathcal{M})$.

Our main result, Theorem 5.4, is as follows. Assume X is local and universally catenary with closed point x_0. Assume the codimension of each irreducible component of the fiber of $\mathrm{Proj}(\mathcal{R}(\mathcal{M}))$ over x_0 is at least 2. Let h be a general element of the maximal ideal of $\mathcal{O}_{X,x_0}$. Then $h \notin \mathrm{z.div}(\mathcal{N}^n/\overline{\mathcal{M}^n})$ for each n. Furthermore, we describe explicitly the genericity conditions on h. In Sects. 5–7 we apply Theorem 5.4 to two seemingly unrelated problems from commutative algebra and algebraic geometry.

First we analyze the set $\bigcup_{n=1}^{\infty} \mathrm{Ass}_X(\mathcal{N}^n/\overline{\mathcal{M}^n})$. It is a hard problem to show that this set is finite, because the modules $\overline{\mathcal{M}^n}$ may not form a finitely generated R-algebra.