Journal of Algebra 508 (2018) 339–363

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Signatures of hermitian forms, positivity, and an answer to a question of Procesi and Schacher

Vincent Astier, Thomas Unger *

School of Mathematics and Statistics, University College Dublin, Belfield, Dublin 4, Ireland

ARTICLE INFO

Article history:
Received 5 January 2017
Available online 7 May 2018
Communicated by Louis Rowen

MSC:
16K20
11E39
13J30

Keywords:
Central simple algebra
Involution
Formally real field
Hermitian form
Signature
Positivity
Sum of hermitian squares

ABSTRACT

Using the theory of signatures of hermitian forms over algebras with involution, developed by us in earlier work, we introduce a notion of positivity for symmetric elements and prove a noncommutative analogue of Artin's solution to Hilbert's 17th problem, characterizing totally positive elements in terms of weighted sums of hermitian squares. As a consequence we obtain an earlier result of Procesi and Schacher and give a complete answer to their question about representation of elements as sums of hermitian squares.

1. Introduction

We use the theory of signatures of hermitian forms, a tool we developed and studied in [4], [5] and [3], to introduce a natural notion of positivity for symmetric elements

* Corresponding author.
E-mail addresses: vincent.astier@ucd.ie (V. Astier), thomas.unger@ucd.ie (T. Unger).

https://doi.org/10.1016/j.jalgebra.2018.05.004
0021-8693/

in an algebra with involution, inspired by the theory of quadratic forms; signatures of one-dimensional hermitian forms over algebras with an involution can take values outside of $\{-1, 1\}$ and it is therefore natural to single out those symmetric elements whose associated hermitian form has maximal signature at a given ordering. We call such elements maximal at the ordering and characterize the elements that are maximal at all orderings in terms of weighted sums of hermitian squares, thus obtaining an analogue of Artin's solution to Hilbert's 17th problem for algebras with involution, cf. Section 3. The proof is obtained via signatures, allowing us to use the hermitian version of Pfister's local-global principle. This provides a short and conceptual argument, based on torsion in the Witt group.

Procesi and Schacher [19] already considered such a noncommutative version of Artin's theorem in this context, using a notion of positivity based on involution trace forms that seems to go back to Albert (e.g. [1]) and Weil [23]. They showed that every totally positive element (in their sense) in an algebra with involution is a sum of squares of symmetric elements, and thus of hermitian squares, with weights, cf. [19, Theorem 5.4]. They also asked if these weights could be removed [19, p. 404]. The answer to this question is in general no, as shown in [12].

Our approach via signatures makes it possible to obtain the sum of hermitian squares version of their theorem as a consequence of Theorem 3.6. It also allows us to single out the set of orderings relevant to their question (the non-nil orderings) and to rephrase it in a natural way, which can then be fully answered (Theorem 4.19).

2. Algebras with involution and signatures of hermitian forms

We present the notation and main tools used in this paper and refer to the standard references [13], [14], [15] and [22] as well as [4] and [5] for the details.

2.1. Algebras with involution, hermitian forms

For a ring A, an involution σ on A and $\varepsilon \in \{-1, 1\}$, we denote the set of ε-symmetric elements of A with respect to σ by $\mathrm{Sym}_\varepsilon(A, \sigma) = \{a \in A \mid \sigma(a) = \varepsilon a\}$. We also denote the set of invertible elements of A by $A^\times$ and let $\mathrm{Sym}_\varepsilon(A, \sigma)^\times := \mathrm{Sym}_\varepsilon(A, \sigma) \cap A^\times$.

Let F be a field of characteristic different from 2. We denote by $W(F)$ the Witt ring of F, by X_F the space of orderings of F, and by F_P a real closure of F at an ordering $P \in X_F$. We allow for the possibility that F is not formally real, i.e. that $X_F = \varnothing$. By an *F-algebra with involution* we mean a pair (A, σ) where A is a finite-dimensional simple F-algebra with centre a field K, equipped with an involution $\sigma : A \to A$, such that $F = K \cap \mathrm{Sym}(A, \sigma)$. Observe that $\dim_F K \leqslant 2$. We say that σ is *of the first kind* if $K = F$ and *of the second kind* (or *of unitary type*) otherwise. Involutions of the first kind can be further subdivided into those of *orthogonal type* and those of *symplectic type*, depending on the dimension of $\mathrm{Sym}(A, \sigma)$. We let $\iota = \sigma|_K$ and note that $\iota = \mathrm{id}_F$ if σ