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Semi-galois categories II: An arithmetic analogue of Christol's theorem

Takeo Uramoto

Graduate School of Information Sciences, Tohoku University, Japan

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ABSTRACT

In connection with our previous work on *semi-galois categories* [1,2], this paper proves an arithmetic analogue of *Christol's theorem* concerning an automata-theoretic characterization of when a formal power series $\xi = \sum \xi_n t^n \in \mathbb{F}_q[[t]]$ over finite field $\mathbb{F}_q$ is algebraic over the polynomial ring $\mathbb{F}_q[t]$. There are by now several variants of Christol's theorem, all of which are concerned with rings of positive characteristic. This paper provides an arithmetic (or $\mathbb{F}_1$-) variant of Christol's theorem in the sense that it replaces the polynomial ring $\mathbb{F}_q[t]$ with the ring O_K of integers of a number field K and the ring $\mathbb{F}_q[[t]]$ of formal power series with the ring of Witt vectors. We also study some related problems.

1. Introduction

The purpose of this paper is to prove an arithmetic analogue of the following theorem due to Christol [3,4]: Let $\mathbb{F}_q$ be the finite field of q elements and denote by $\mathbb{F}_q[[t]]$ the ring of formal power series over $\mathbb{F}_q$; also by $\mathbb{F}_q[t] \subseteq \mathbb{F}_q[[t]]$ the polynomial ring. Then *Christol's theorem* claims as follows (see §3.1 for undefined terminology here):

E-mail address: takeo.uramoto@tohoku.ac.jp.

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Theorem 1.1 *(Christol's theorem). A formal power series* $\xi = \sum \xi_n t^n \in \mathbb{F}_q[[t]]$ *is algebraic over* $\mathbb{F}_q[t]$ *if and only if the coefficients* $(\xi_n) \in \mathbb{F}_q^{\mathbb{N}_{\geq 0}}$ *can be generated by some deterministic finite automaton (cf. §3.1).*

This theorem lies at the intersection of number theory and automata theory; in the number theoretic context, automata theory typically has provided ideas to classify number-theoretic objects (such as formal power series [3–5], or expansions of real numbers by integer base [6–8] and continued fraction [9]) from the viewpoint of computational complexity: Automata theorists ask how computationally complex it is to produce sequences of coefficients of formal power series or expansions of real numbers; and measure their complexity in terms of hierarchies of computational models that generate them. Christol's theorem is one of pioneering results of this direction, which characterized algebraicity of formal power series over $\mathbb{F}_q[t]$ in terms of *deterministic finite automata* — the simplest computational models among others (such as Turing machines); this theorem provided a reasonable criterion to study transcendence of formal power series [10].

Several variants of Christol's theorem have been investigated in the literature, all of which were concerned with rings of positive characteristic. (See [11] for more information on Christol's theorem and its variants.) The current paper is concerned with proving an *arithmetic (or* $\mathbb{F}_1$*-) analogue* of Christol's theorem in the sense that the polynomial ring $\mathbb{F}_q[t]$ is replaced with the ring O_K of integers of a number field K; in this variant, the ring $\mathbb{F}_q[[t]]$ of formal power series is replaced with the ring of *Witt vectors* [12] (cf. §2). More formally, our variant of Christol's theorem claims as follows (see §3.2 for undefined terminology here):

Theorem 1.2 *(Arithmetic analogue of Christol's theorem). A Witt vector* $\xi = (\xi_{\mathfrak{a}}) \in W_{O_K}(O_{\bar{K}})$ *is integral over* O_K *if and only if its coefficients* $(\xi_{\mathfrak{a}}) \in O_{\bar{K}}^{I_K}$ *can be generated by some deterministic finite automaton (cf. §3.2).*

As in the case of Christol's original theorem, the technical core of this proof is to estimate the size of the orbits of integral Witt vectors ξ under the actions of (infinitely many) *Frobenius liftings* $\psi_{\mathfrak{p}}$ that are canonically equipped to $W_{O_K}(O_{\bar{K}})$ as it forms a Λ-*ring* (cf. §2). For effective estimation, we will develop in §3 several basic bounds on coefficients of Witt vectors. In the same way as original one, our arithmetic analogue of Christol theorem relates the integrality of Witt vectors $\xi \in W_{O_K}(O_{\bar{K}})$ over O_K and the automata-theoretic complexity of their coefficients; we say that a Witt vector $\xi = (\xi_{\mathfrak{a}})$ is *automatic* if its coefficients $\xi_{\mathfrak{a}}$ can be generated by some finite automaton.

After this proof, we combine our Christol theorem with a strong classification result of certain Λ-rings due to Borger and de Smit [13,14], to conclude an explicit description of the integral closure of O_K within $W_{O_K}(O_{\bar{K}})$ (§4). To be specific, Borger and de Smit proved in [14] that the category $\mathscr{C}_K$ of those Λ-rings which are finite étale over K and have *integral models* (cf. §2) is dually equivalent to the category $\mathscr{B}_f DR_K$ of *finite* DR_K*-sets*, i.e. finite sets equipped with continuous actions of the profinite monoid DR_K