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Finite Gröbner basis algebras with unsolvable nilpotency problem and zero divisors problem [☆]

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ABSTRACT

This work presents a sample construction of two algebras both with the ideal of relations defined by a finite Gröbner basis. For the first algebra the question whether a given element is nilpotent is algorithmically unsolvable, for the second one the question whether a given element is a zero divisor is algorithmically unsolvable. This gives a negative answer to questions raised by Latyshev.

1. Introduction

The word equality problem in finitely presented semigroups (and in algebras) cannot be algorithmically solved. This was proved in 1947 by Markov ([13]) and independently

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by Post ([17]). In 1952 Novikov constructed the first example of the group with unsolvable problem of word equality (see [15] and [16]).

In 1962 Shirshov proved solvability of the equality problem for Lie algebras with one relation and raised a question about finitely defined Lie algebras (see [22]).

In 1972 Bokut settled this problem. In particular, he showed the existence of a finitely defined Lie algebra over an arbitrary field with algorithmically unsolvable identity problem ([2]).

A detailed overview of algorithmically unsolvable problems can be found in [4].

Otherwise, some problems become decidable if a finite Gröbner basis defines a relations ideal. In this case it is easy to determine whether two elements of the algebra are equal or not (see [1]).

Gröbner bases for various structures are investigated by the Bokut school in Guangzhou ([3]).

In his work, Piontkovsky extended the concept of obstruction, introduced by Latyshev (see [18], [19], [20], [21]).

Latyshev raised the question concerning the existence of an algorithm that can find out if a given element is either a zero divisor or a nilpotent element when the ideal of relations in the algebra is defined by a finite Gröbner basis.

Similar questions for monomial automaton algebras can be solved. In this case the existence of an algorithm for nilpotent element or a zero divisor was proved by Kanel-Belov, Borisenko and Latyshev [11]. Note that these algebras are not Noetherian and not weak Noetherian. Iyudu showed that the element property of being one-sided zero divisor is recognizable in the class of algebras with a one-sided limited processing (see [6], [7]). It also follows from a solvability of a linear recurrence relations system on a tree (see [9]).

Algorithmic questions in algebras with finite Gröbner basis are considered in [5].

An example of an algebra with a finite Gröbner basis and algorithmically unsolvable problem of zero divisor is constructed in [8]. However one should consider there a semi-DEGLEX order, where the letter t has weight 16, and all other letters have weight 1. This is a reduction order, and relations (01)–(25) in [8, Proposition 3.5] form a finite Gröbner basis, where left-hand sides are larger than right-hand sides.

A notion of *Gröbner basis* (better to say *Gröbner–Shirshov basis*) first appeared in the context of noncommutative (and not Noetherian) algebra. Note also that Poincaré–Birkhoff–Witt theorem can be canonically proved using Gröbner bases. For more detailed discussions of these questions see in [2], [23], [10], [11], [12].

In the present paper we construct an algebra with a finite Gröbner basis and algorithmically unsolvable problem of nilpotency. We also provide a shorter construction for the zero divisors question.

For these constructions we simulate a universal Turing machine, each step of which corresponds to a multiplication from the left by a chosen letter.

Thus, to determine whether an element is a zero divisor or is a nilpotent, it is not enough for an algebra to have a finite Gröbner basis.

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