Journal of Algebra 507 (2018) 1–18

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On derived Hall numbers for tame quivers

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ARTICLE INFO

Article history:
Received 15 November 2016
Available online 9 April 2018
Communicated by Leonard L. Scott, Jr.

MSC:
16G20
17B20
17B37

Keywords:
Derived Hall algebra
Derived Hall number
Generic function
Tame quiver

ABSTRACT

In the present paper we study the derived Hall algebra for the bounded derived category of the finite dimensional nilpotent representations of a Dynkin or tame quiver over a finite field. We show that for any three given objects in the bounded derived category, the associated derived Hall numbers are given by a rational function in the cardinalities of ground fields.

1. Introduction

In 1990, Ringel [17] introduced the Hall algebra $\mathcal{H}(A)$ of a finite dimensional algebra A over a finite field. By definition, the Hall algebra $\mathcal{H}(A)$ is a free abelian group with basis the isoclasses (isomorphism classes) of finite dimensional A-modules, and the structure

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[1] The authors are supported by the Natural Science Foundation of China (Grant No. 11471269; No. 211070B31704).

https://doi.org/10.1016/j.jalgebra.2018.04.005

constants are given by the so-called Hall numbers, which count the number of certain submodules. Ringel [19] proved that if A is representation finite and hereditary, then the twisted Hall algebra $\mathcal{H}_v(A)$, called the Ringel–Hall algebra, is isomorphic to the positive part of the corresponding quantized enveloping algebra. Later on, Green [5] introduced a bialgebra structure on $\mathcal{H}_v(A)$ for each hereditary algebra A, and he showed that the composition subalgebra of $\mathcal{H}_v(A)$ generated by simple A-modules provides a realization of the positive part of the corresponding quantized enveloping algebra.

In case that A is representation finite and hereditary, Ringel [18] showed that the Hall numbers of A are actually integer polynomials in the cardinalities of finite fields. These polynomials are called Hall polynomials as in the classical case; see [12]. Then one can define the generic Hall algebra $H_{\boldsymbol{v}}(A)$ over the Laurent polynomial ring $\mathbb{Z}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ and its degeneration $H_1(A)$ at $\boldsymbol{v} = 1$. It was shown by Ringel [17,18] that $H_1(A) \otimes \mathbb{C}$ is isomorphic to the positive part of the universal enveloping algebra of the complex semisimple Lie algebra associated with A. Since then, much subsequent work was devoted to the study of Hall polynomials for various classes of algebras. In [20], Ringel conjectured that Hall polynomials exist for all representation-finite algebras. This conjecture has been proved only for some special algebras, see for example [4,6,7,13,14,17]. By Ringel [21], Hall polynomials exist in the category of finite dimensional nilpotent representations for a cyclic quiver. Some Hall polynomials for representations of the Kronecker quiver has also been calculated in [26,23]. More generally, Hubery [9] proved that Hall polynomials exist for all tame (affine) quivers with respect to the decomposition classes of Bongartz and Dudek (cf. [1]). Recently, Deng and Ruan [3] have generalized Hubery's result to a more general setting. They proved that Hall polynomials exist for domestic weighted projective lines with respect to the decomposition sequences. As an application, they obtained the existence of Hall polynomials for tame quivers. It is worth mentioning that the existence of Hall polynomials has gained importance by the relevance of quiver Grassmannians with cluster algebras (cf. [2]).

In order to realize the entire quantized enveloping algebra via Hall algebra approach, one turns to consider the Hall algebras of triangulated categories (cf. [10,24,25]). Toën [24] defined a derived Hall algebra for a differential graded category satisfying some finiteness conditions, where the structure constants are given by the so-called derived Hall numbers. Later on, Xiao and Xu [25] investigated the derived Hall algebra for an arbitrary triangulated category under certain finiteness conditions, which includes the bounded derived category of a finite dimensional hereditary algebra over a finite field.

A natural question is whether the derived Hall numbers also have a generic phenomenon like Hall polynomials for Hall numbers. The aim of this paper is to give a positive answer to this question for Dynkin and tame quivers. In the Ringel–Hall algebra case, for Dynkin quiver, Ringel [18] proved the existence of Hall polynomials by using the associativity formula and the directedness of Auslander–Reiten quiver. In [9], Hubery succeeded in applying Green's Formula to the study of Hall polynomials for Dynkin and tame quivers, which reduces the problem to a situation where the result is clear, by induction on the dimension vectors of the middle terms. But for the derived Hall algebra