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Quiver-theoretical approach to dynamical Yang–Baxter maps ☆

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A B S T R A C T

A dynamical Yang–Baxter map, introduced by Shibukawa, is a solution of the set-theoretical analogue of the dynamical Yang–Baxter equation. In this paper, we initiate a quiver-theoretical approach for the study of dynamical Yang–Baxter maps. Our key observation is that the category of dynamical sets over a set Λ, introduced by Shibukawa to establish a categorical framework to deal with dynamical Yang–Baxter maps, can be embedded into the category of quivers with vertices Λ. By using this embedding, we shed light on Shibukawa's classification result of a certain class of dynamical Yang–Baxter maps and extend his construction to obtain a new class of dynamical Yang–Baxter maps. We also discuss a relation between Shibukawa's bialgebroid associated to a dynamical Yang–Baxter map and Hayashi's weak bialgebra associated to a star-triangular face model.

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1. Introduction

The Yang–Baxter equation was first considered independently by McGuire [16] and Yang [26] in their study of one-dimensional many body problems. Finding a (constant) solution of the equation is equivalent to solving the equation

$$(\sigma \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \sigma)(\mathrm{id}_V \otimes \sigma) = (\mathrm{id}_V \otimes \sigma)(\sigma \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \sigma) \tag{1.1}$$

for a linear operator $\sigma : V \otimes V \to V \otimes V$ on the two-fold tensor product of a vector space V. Thus, abusing terminology, we refer to (1.1) as the Yang–Baxter equation in this paper. As a natural analogue of (1.1), Drinfeld [3] proposed to investigate the set-theoretical Yang–Baxter equation

$$(\sigma \times \mathrm{id}_X)(\mathrm{id}_X \times \sigma)(\mathrm{id}_X \times \sigma) = (\mathrm{id}_X \times \sigma)(\sigma \times \mathrm{id}_X)(\mathrm{id}_X \times \sigma) \tag{1.2}$$

for a map $\sigma : X \times X \to X \times X$ on the two-fold Cartesian product of a set X. Solutions of (1.2), called *Yang–Baxter maps*, have been studied as well as the solutions of the ordinary Yang–Baxter equation.

Gervais and Neveu [7] have introduced a generalization of the Yang–Baxter equation (1.2), called the *dynamical Yang–Baxter equation*. Let $\mathfrak{h}$ be a commutative Lie algebra, and let V be a diagonalizable $\mathfrak{h}$-module. The dynamical Yang–Baxter equation on V is (equivalent to) the equation

$$\sigma(\lambda)_{12} \circ \sigma(\lambda - h^{(1)})_{23} \circ \sigma(\lambda)_{12} = \sigma(\lambda - h^{(1)})_{23} \circ \sigma(\lambda)_{12} \circ \sigma(\lambda - h^{(1)})_{23} \tag{1.3}$$

for a family $\{\sigma(\lambda) \in \mathrm{End}_{\mathfrak{h}}(V \otimes V)\}_{\lambda \in \mathfrak{h}^*}$ of $\mathfrak{h}$-equivariant linear operators. Here, $\sigma(\lambda)_{12}$ and $\sigma(\lambda - h^{(1)})_{23}$ are linear operators on $V \otimes V \otimes V$ defined by

$$\sigma(\lambda)_{12}(v_1 \otimes v_2 \otimes v_3) = \sigma(\lambda)(v_1 \otimes v_2) \otimes v_3,$$
$$\sigma(\lambda - h^{(1)})_{23}(v_1 \otimes v_2 \otimes v_3) = v_1 \otimes \sigma(\lambda - \mathrm{wt}(v_1))(v_2 \otimes v_3)$$

for $\lambda \in \mathfrak{h}$ and weight vectors $v_1, v_2, v_3 \in V$. Felder [6] studied mathematical aspects of this equation. Motivated by its relation to conformal field theory and statistical mechanics, the dynamical Yang–Baxter equation also have been studied extensively; see, *e.g.*, the lecture book of Etingof–Latour [4] and references therein.

It is interesting to study a set-theoretical analogue — like (1.2) — of the dynamical Yang–Baxter equation (1.3). Shibukawa gave a mathematical formulation of such an equation: Let Λ be a non-empty set, and let X be a set equipped with a map $\Lambda \times X \to \Lambda$ expressed as $(\lambda, x) \mapsto \lambda \triangleleft x$. The *set-theoretical dynamical Yang–Baxter equation* [19–21] is the equation

$$\sigma(\lambda)_{12} \circ \sigma(\lambda \triangleleft X^{(1)})_{23} \circ \sigma(\lambda)_{12} = \sigma(\lambda \triangleleft X^{(1)})_{23} \circ \sigma(\lambda)_{12} \circ \sigma(\lambda \triangleleft X^{(1)})_{23} \tag{1.4}$$