Journal of Algebra 507 (2018) 103–119

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On the dimension of polynomial semirings

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ARTICLE INFO

Article history:
Received 12 November 2015
Available online 10 April 2018
Communicated by Louis Rowen

MSC:
primary 16Y60, 12K10
secondary 14T05, 06F05

Keywords:
Idempotent semirings
Idempotent semifields
Polynomial semirings
Krull dimension

ABSTRACT

In our previous work, motivated by the study of tropical polynomials, a definition for prime congruences was given for an arbitrary commutative semiring. It was shown that for additively idempotent semirings this class exhibits some analogous properties to prime ideals in ring theory. The current paper focuses on the resulting notion of Krull dimension, which is defined as the length of the longest chain of prime congruences. Our main result states that for any additively idempotent semiring A, the semiring of polynomials $\dim A[x_1, \ldots, x_n]$ and the semiring of Laurent polynomials $A[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, we have $\dim A[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = \dim A[x_1, \ldots, x_n] = \dim A + n$.

1. Introduction

In [4] we introduced a notion of dimension for arbitrary commutative semirings, which is defined in terms of chains of congruences and extends the usual notion of Krull di-

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[1] The first author is partially supported by the grants of the National Research, Development and Innovation Office, NKFI PD-121410 and NKFI K-119934.

https://doi.org/10.1016/j.jalgebra.2018.04.007
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mension of commutative rings. The current work focuses on determining the dimension of additively idempotent polynomial semirings. One motivation to study this class is provided by tropical geometry. There, from an algebraic point of view, one is interested in the properties of polynomial semirings over the tropical max-plus semifield $\mathbb{T} = \mathbb{R}_{max}$. A different motivation comes from the semiring approach to characteristic 1 geometry, where the semifield of integers $\mathbb{Z}_{max} \subset \mathbb{T}$ and the two element additively idempotent semifield $\mathbb{B}$ play an important role (cf. [1]).

In ring theory congruences, or equivalently homomorphisms, are determined by the ideal that is their kernel (i.e. the equivalence class of the 0 element). From this perspective semirings behave quite differently: in general the kernel of a congruence contains very little information about which elements are identified. In fact one can easily find examples of commutative semirings with a complicated lattice of congruences all of which have trivial kernels. With this consideration in mind, in [4] the notion of primeness is extended to the congruences of a general commutative semiring, and the class of prime congruences is described in the polynomial (and Laurent polynomial) semirings over the semifields $\mathbb{B}$, $\mathbb{T}$ and $\mathbb{Z}_{max}$. The key application of this theory in [4] is to prove a tropical Nullstellensatz. However, it is also observed that the prime congruences yield a notion of Krull dimension which behaves intuitively in the sense that an n variable polynomial ring over $\mathbb{T}, \mathbb{Z}_{max}$ or $\mathbb{B}$ always has dimension n larger than that of its ground semifield. The aim of the current paper is to generalize this result for the polynomial and Laurent polynomial semirings over arbitrary additively idempotent commutative semirings, which we will refer to as $\mathbb{B}$-algebras.

A different approach to establish a tropical notion of dimension using chains of congruences was taken by L. Rowen and T. Perri in [5], which we will briefly recall in Remark 2.12. A common theme in [5] and the present work is that to obtain a good notion of dimension one has to bypass the difficulties that come from polynomial semirings having too many congruences. In fact, as we will see in Proposition 2.15, any additively idempotent polynomial semiring in at least 2 variables has infinite chains of congruences with cancellative quotients.

The main result of the current work is Theorem 3.18, which concerns the polynomial semiring $A[x]$ and the Laurent polynomial semiring $A[x^{\pm 1}]$ over an arbitrary additively idempotent commutative semiring A.

Theorem 1.1. *Let A be a $\mathbb{B}$-algebra with* $\dim A < \infty$. *Then we have that*

$$\dim A[x_1^{\pm 1}, \dots, x_n^{\pm 1}] = \dim A[x_1, \dots, x_n] = \dim A + n.$$

When comparing to the classical ring theoretic setting, it is somewhat surprising that Theorem 3.18 holds without any restriction on A. In commutative ring theory, for any Noetherian ring R $\dim R[x] = \dim R + 1$ (see for example [2]). However, when the Noetherian restriction is dropped $\dim R[x]$ can be any integer between $\dim R + 1$ and $2 \dim R + 1$ (see [6]).