Journal of Algebra 507 (2018) 186–199

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Correspondence of models under local theta lifting

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A R T I C L E   I N F O

Article history:
Received 8 August 2017
Available online 17 April 2018
Communicated by Vera Serganova

Keywords:
Local theta correspondence
Model transition
Generalized Shalika models

A B S T R A C T

We study correspondence of models of representations under local theta lifting between general linear groups and unitary groups over a non-archimedean local field of characteristic 0.

1. Introduction

Denote by F a non-Archimedean local field of characteristic 0. When we treat representations of a group G defined over F, representations will always be admissible representations.

Definition 1. Let π be an irreducible representation of G.

(i) Let H be a closed subgroup of G and Ψ a character on H. An irreducible representation π of G is said to have *a model associated to* (H, Ψ) if $\mathrm{Hom}_H(\pi, \Psi) \neq 0$.

(ii) The model for π associated to (H, Ψ) is *unique* (or *multiplicity one theorem holds*) if

$$\dim \mathrm{Hom}_H(\pi, \Psi) \leq 1.$$

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https://doi.org/10.1016/j.jalgebra.2018.04.010

Let $G = \mathrm{GL}_{2n}(F)$ and let π be an irreducible representation of G. Jacquet and Shalika [7] introduced the notion of *Shalika model* for automorphic cuspidal representations of GL_{2n} and found a relation between this model and an analytic property of an exterior square partial L-function. The local counterpart of this relation was studied by Jiang, Nien and Qin;

Theorem 1. *([4] Theorem 1.1) Assume that π is supercuspidal. The exterior square local L-factor $L(s, \pi, \wedge^2)$ has a pole at $s = 0$ if and only if π has a nonzero Shalika model.*

This theorem was extended to the case of discrete series representations by [10].

Uniqueness of Shalika model was first proven by [6]. Their strategy is as follows; they showed that if π has a nonzero Shalika model, then π has a model associated to $(\mathrm{GL}_n(F) \times \mathrm{GL}_n(F), 1)$. Once this claim is proved, the problem is reduced to prove that the model associated to $(\mathrm{GL}_n(F) \times \mathrm{GL}_n(F), 1)$ is unique.

Definition 2. Let G be a group and let H be its subgroup. The pair (G, H) is called *a Gelfand pair* if the model for representations of G associated to $(H, 1)$ is unique, where 1 denotes the trivial character on H.

They actually proved that the pair $(\mathrm{GL}_{p+q}, \mathrm{GL}_p \times \mathrm{GL}_q)$ is a Gelfand pair for any positive integer p, q. Nien found a more direct proof for uniqueness of Shalika models [14].

Shalika model is one of the degenerate Whittaker models associated to a Siegel parabolic subgroup. From this perspective, it is natural to consider analogous models for representations of other classical groups. In [5], Jiang and Qin introduced the notion of *generalized Shalika model* for representations of split $\mathrm{SO}_{4n}(F)$. Uniqueness for this model was also proved by [13] in an analogous manner to that for Shalika models.

Jiang, Nien and Qin conjectured correspondence between generalized Shalika model for representations of $\mathrm{SO}_{4n}(F)$ and certain linear model for representations of $\mathrm{Sp}_{4n}(F)$ (see [3] section 4 p. 542).

The correspondence from generalized Shalika models to linear models was proved independently by [2] and [9]. Note that Liu considered in more general setting than Hanzer and proved similar correspondence for not necessarily equal rank dual pairs. Liu also defined generalized Shalika model for representations for $\mathrm{Sp}_{4n}(F)$ and showed similar correspondence in one direction under theta lifts from $\mathrm{Sp}_{4n}(F)$ to $\mathrm{O}_{4m}(F)$ with $2m \geq n$. Exactly the same argument shows that similar correspondence from $\mathrm{Sp}_{2n}(F)$ to $\mathrm{O}_m(F)$, i.e. if an irreducible representation π of $\mathrm{Sp}_{2n}(F)$ has a nonzero generalized Shalika model and its theta lift π' to $\mathrm{O}_m(F)$, $m \geq n$ is nonzero, then π' has a model associated to $(\mathrm{O}_n(F) \times \mathrm{O}_{m-n}(F), 1)$. The symmetric pair $(\mathrm{O}_m(F), \mathrm{O}_n(F) \times \mathrm{O}_{m-n}(F))$ is a Gelfand pair if $m = n + 1$ (see [9] Theorem 7.2.7 or [1] Theorem 1'). Therefore, uniqueness of generalized Shalika models for representations of $\mathrm{Sp}_{2n}(F)$ holds under the condition on non-vanishing of local theta lifts.