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Comparison of Steinberg modules for a field and a subfield

Avner Ash

Boston College, Chestnut Hill, MA 02467, United States of America

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A B S T R A C T

Let E/F be an extension of fields. We investigate the Steinberg module of $GL(n,E)$ restricted to $GL(n,F)$. We give examples for $n = 2$ and $n = 3$ where we compute the homology of a subgroup of $GL(n,F)$ with coefficients in the Steinberg module of $GL(n,E)$.

1. Introduction

The Steinberg module $St(G(F))$ of the F-points of a reductive algebraic group G over a field F plays an important role in representation theory and the cohomology of arithmetic groups. For example, if F is a number field and Γ is an arithmetic subgroup of $G(F)$ whose torsion primes are invertible on a Γ-module M, then the Borel–Serre duality theorem [3], as improved on page 280 of [4], says that the homology $H_i(\Gamma, St(G(F)) \otimes M)$ is isomorphic to the cohomology $H^{d-i}(\Gamma, M)$ where d is the virtual cohomological dimension of Γ.

E-mail address: Avner.Ash@bc.edu.

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A natural question, which appears not to have been asked in the literature, is "what is $H_i(\Gamma, St(G(E)) \otimes M)$ if E is an extension field of F?" This leads to a prior question: What can we say about the restriction of the $G(E)$-module $St(G(E))$ to $G(F)$?

This paper deals with these questions in the case when $G = GL(n)$, F is any field and E/F is any extension. We work with $GL(n)$ because for this group we have a very explicit description of its Steinberg module which allows us to compare $St(GL(n, F))$ and $St(GL(n, E))$ by means of a certain filtration of the latter.

Fix a ring k as a base ring for the coefficients of St. In place of $St(GL(n, K))$ we write $St(K^n)$. We define a filtration on $St(E^n)$ which is stable under $GL(n, F)$. The associated graded module of this filtration consists of $St(F^n)$ plus a direct sum of induced representations from proper subgroups of $GL(n, F)$.

Our main results are Theorems 3.6, 3.9, 4.6, and 4.7. Because they involve some technical notions, we will not state them in this introduction. Here, however, we can give their flavor by stating a corollary to them which we use in the last two sections, where we work out examples for $n = 2$ and $n = 3$. This is Theorem 4.9, here restated without the technical terminology and with a slight alteration of the notation:

Theorem 1.1. *Let E/F be a nontrivial extension of fields, $n \geq 1$. Then there exists a finite increasing filtration $0 = \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_n = St(E^n)$ by $GL(n, F)$-modules with the following property:*

Suppose given a subgroup $\Gamma \subset GL(n, F)$ and an integer $m \in [0, n]$, and assume that for each E-subspace Y of E^n of dimension m such that $Y \cap E^n = \{0\}$, there is an $e \in E^n$ such that

(1) *for any $\gamma \in Stab_\Gamma(Y)$, $\gamma e \in Ee + Y$;*
(2) *if A_i are F-subspaces of dimension $n - m - 1$ of F^n such that $E^n = Ee + EA_i + Y$ for $i = 1, 2$ and if $EA_1 + Y = EA_2 + Y$, then $A_1 = A_2$.*

For each such Y and E-hyperplane H of V that does not contain e let $A_{Y,H}$ be the unique F-subspace of F^n such that $H = EA_{Y,H} + Y$. Then $\mathcal{F}_m/\mathcal{F}_{m-1}$ is isomorphic as a Γ-module to a direct sum of induced modules

$$\mathrm{Ind}(Stab_\Gamma(Y) \cap Stab_\Gamma(H), \Gamma, St(A_{Y,H}) \otimes_k St(Y)),$$

where Y ranges over a certain subset of such Y's and H ranges over a certain subset of such H's.

In general, we can investigate $H_i(\Gamma, St(E^n) \otimes M)$ using a spectral sequence coming from the filtration. We use Shapiro's lemma to compute the homology of the induced representations. For simplicity, in this paper we let M be the trivial module. We give examples in the last two sections where $H_i(\Gamma, St(E))$ is not finitely generated over k although $H_i(\Gamma, St(F))$ is so. A priori, one might have thought that $H_i(\Gamma, St(E^n))$ would