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Fixed points of endomorphisms of complex tori ☆

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A B S T R A C T

We study the asymptotic behavior of the cardinality of the fixed point set of iterates of an endomorphism of a complex torus. We show that there are precisely three types of behavior of this function: it is either an exponentially growing function, a periodic function, or a product of both.

1. Introduction

Let X be a complex manifold and let $f : X \to X$ be a holomorphic map. It is a natural question to ask how many fixed points f has, and what geometric and topological information they can give us about X. The answer to this question of course depends on X and the map chosen, and as stated the question is too broad to be useful. As noted in the introduction of [2], one expects a more uniform answer by looking at asymptotic properties of fixed points of iterates of f, as this point of view has been successfully

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adopted in several areas of complex and algebraic geometry (see [2] and the references cited in the introduction). Let

$$F_f(n) = F(n) := \#\mathrm{Fix}(f^n)$$

denote the number of fixed points of f^n if this number is finite, and 0 if not.

We are interested in studying the asymptotic behavior of $F(n)$ in the case that X is a complex torus. In this case, the Lefschetz Fixed-Point Theorem for complex tori gives the exact number of fixed points of f in terms of the eigenvalues of the action of f on the tangent space at the origin. Indeed, let $\rho_a(f) \in \mathrm{End}(T_X(0))$ denote the differential of f at 0 (i.e. the analytic representation of f), and let $\lambda_1, \ldots, \lambda_g$ be its eigenvalues. Then by [3, 13.1.2],

$$\#\mathrm{Fix}(f^n) = \left| \prod_{i=1}^{g} (1 - \lambda_i^n) \right|^2 . \tag{1}$$

This shows that the asymptotic behavior of $F(n)$ is governed by the eigenvalues of $\rho_a(f)$. This is not surprising, given the fact that the eigenvalues of $\rho_a(f)$ control many of the topological properties of the dynamical system induced by f (such as its topological entropy; see the next section for details).

Our motivation for studying this problem for complex tori comes from [2], where the authors give a complete classification of the behavior of $F(n)$ in the case that X is a two-dimensional complex torus. Indeed, it is shown that on a two-dimensional complex torus, $F(n)$ either

- grows exponentially
- is periodic
- is a product of these two behaviors.

One of the key lemmas that is used in this classification is the fact that if λ is an eigenvalue of an endomorphism of a complex two-dimensional torus and $|\lambda| = 1$, then λ must be a root of unity. This is no longer true for higher dimensions, and this, a priori, produces one of the main difficulties for extending the results found in dimension two.

It seems that the appearance of eigenvalues that lie on the unit circle and are not roots of unity should be interesting in their own right. For example, in [8], Oguiso studies Salem numbers that appear as eigenvalues of endomorphisms of complex tori in the context of studying the existence of equivariant holomorphic fibrations between tori. For this reason, we briefly study properties of endomorphisms with these types of eigenvalues, as well as find examples of when these eigenvalues appear.

Although interesting eigenvalues appear for higher dimensional complex tori, our main theorem shows that the asymptotic behavior of $F(n)$ is virtually the same as in dimension two. Indeed, we prove: