ARTICLE IN PRESS YJABR:16602

Journal of Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Quantitative aspects of the generalized differential Lüroth's Theorem ☆

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ARTICLE INFO

Article history:
Received 25 January 2017
Available online xxxx
Communicated by Bruno Salvy

MSC:
12Y05
12H05

Keywords:
Differential algebra
Lüroth's Theorem
Differentiation index

ABSTRACT

Let $\mathcal{F}$ be a differential field of characteristic 0, $\mathbf{t} = t_1, \dots, t_m$ a finite set of differential indeterminates over $\mathcal{F}$ and $\mathcal{G} \subset \mathcal{F}\langle \mathbf{t} \rangle$ a differential field extension of $\mathcal{F}$, generated by nonconstant rational functions $\alpha_1, \dots, \alpha_n$ of total degree and order bounded by d and $e \geq 1$ respectively. The generalized differential Lüroth's Theorem states that if the differential transcendence degree of $\mathcal{G}$ over $\mathcal{F}$ is 1, there exists $v \in \mathcal{G}$ such that $\mathcal{G} = \mathcal{F}\langle v \rangle$. We prove a new explicit upper bound for the degree of v in terms of n, m, d and e. Further, we exhibit an effective procedure to compute v.

☆ Partially supported by Universidad de Buenos Aires: UBACYT 20020120100133BA (2013–2016) and UBACYT 20020160100039BA (2017).

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https://doi.org/10.1016/j.jalgebra.2018.01.050
0021-8693/

1. Introduction

Let $\mathcal{F}$ be a differential field, $\mathbf{t} = t_1, \dots, t_m$ a finite set of differential indeterminates over $\mathcal{F}$ and $\mathcal{G} \subset \mathcal{F}\langle \mathbf{t} \rangle$ a differential field extension of $\mathcal{F}$. The *generalized differential Lüroth's Theorem* states that *if the differential transcendence degree of* $\mathcal{G}$ *over* $\mathcal{F}$ *is* 1, *then there exists* $v \in \mathcal{G}$ *such that* $\mathcal{G} = \mathcal{F}\langle v \rangle$. The case $m = 1$ is the classical differential Lüroth's Theorem.

This theorem is a differential generalization of similar results valid in the *algebraic* framework: the classical *algebraic* result (i.e. for $m = 1$) is established by Lüroth in 1876 (see [16]). Its extension for an arbitrary integer m in the case of characteristic 0 is given by Gordan ten years later in [7] and in 1951, Igusa [10] shows the generalization for arbitrary characteristic.

In the differential setting, the first version of the classical *differential* Lüroth's Theorem is given by Ritt in 1932 (see [17]) for the field of complex meromorphic functions and extended by Kolchin in [12] and [13] for any differential base field of characteristic 0. Moreover, in his book [11] Kolchin points out that the arguments given by Ritt and himself could be adapted in order to prove the *generalized* differential Lüroth's Theorem for an arbitrary integer m (see [11, Ch. IV, Section 7, Exercise 2]).

The present paper deals with quantitative aspects of the generalized differential Lüroth's Theorem. More precisely, suppose that the intermediate field $\mathcal{G}$ is finitely generated over $\mathcal{F}$ by differential nonconstant rational functions $\alpha_1, \dots, \alpha_n \in \mathcal{F}\langle \mathbf{t} \rangle$ whose total degrees are bounded by an integer d (the total degree of a rational function is defined as the maximum of the total degrees of the numerator and the denominator in an irreducible representation). Let e be an upper bound for the order of $\alpha_1, \dots, \alpha_n$, which for technical reasons we assume to be at least 1. We are interested in the determination of *a priori* bounds for the order and the degree of a *Lüroth generator* v in terms of the parameters m, n, d, e and in the design of an effective method to find v.

An elementary calculation shows that the order of any Lüroth generator is bounded by the minimum of the orders of the generators α_j (see for instance [3, Proposition 5]).

Obtaining an upper bound for the degrees of the polynomials describing the Lüroth generator is a more delicate task. The first results for the case $m = 1$ are given in [3] and [4]. In the present paper, we get new bounds for arbitrary m by suitably adapting and extending the main arguments in our previous paper [3]: in Theorem 17 below, we show that any Lüroth generator v of $\mathcal{G}$ over $\mathcal{F}$ has total degree bounded by

$$\min\left\{ \big((d+1)\big((n+\mu-1)d+1\big)\big)^{\min\{m,2\}e+1} \,,\ (d+1)^{n(\min\{m,n\}e+1)} \right\},$$

where μ is the minimal order of any variable appearing in $\alpha_1, \dots, \alpha_n$. Moreover, if the ground differential field $\mathcal{F}$ contains a nonconstant element, we obtain the better bound

$$\min\left\{ \big((d+1)((n-1)d+1)\big)^{\min\{m,2\}e+1} \,,\ (d+1)^{n(\min\{m,n\}e+1)} \right\}.$$

Please cite this article in press as: L. D'Alfonso et al., Quantitative aspects of the generalized differential Lüroth's Theorem, J. Algebra (2018), https://doi.org/10.1016/j.jalgebra.2018.01.050