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Gradings on classical central simple real Lie algebras

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For any abelian group G, we classify up to isomorphism all G-gradings on the classical central simple Lie algebras, except those of type D_4, over the field of real numbers (or any real closed field).

1. Introduction

Let $\mathcal{R}$ be an algebra (not necessarily associative) over a field and let G be a group. A *G-grading* on $\mathcal{R}$ is a vector space decomposition $\Gamma : \mathcal{R} = \bigoplus_{g\in G} \mathcal{R}_g$ such that

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$\mathcal{R}_g\mathcal{R}_h \subseteq \mathcal{R}_{gh}$ for all $g, h \in G$. The nonzero elements $x \in \mathcal{R}_g$ are said to be *homogeneous of degree* g, which can be written as $\deg x = g$, and the *support* of the grading Γ (or of the graded algebra $\mathcal{R}$) is the set $\{g \in G \mid \mathcal{R}_g \neq 0\}$. The algebra $\mathcal{R}$ may have some additional structure, for example, an involution φ, in which case Γ is required to respect this structure: $\varphi(\mathcal{R}_g) = \mathcal{R}_g$ for all $g \in G$.

Group gradings have been extensively studied for many types of algebras — associative, Lie, Jordan, composition, etc. (see e.g. the recent monograph [12] and the references therein). In the case of gradings on simple Lie algebras, the support generates an abelian subgroup of G, so it is no loss of generality to assume G abelian. We will do so for all gradings considered in this paper.

The classification of fine gradings (up to equivalence) on all finite-dimensional simple Lie algebras over an algebraically closed field of characteristic 0 has recently been completed by the efforts of many authors: see [10], [12, Chapters 3–6], [9], [20] and [11]. The classification of all G-gradings (up to isomorphism) is also known for these algebras, except for types E_6, E_7 and E_8, over an algebraically closed field of characteristic different from 2: see [2], [12, Chapters 3–6] and [13].

On the other hand, group gradings on algebras over the field of real numbers have not yet been sufficiently studied. Fine gradings on real forms of the classical simple complex Lie algebras except D_4 were described in [15], but the equivalence problem remains open. Fine gradings on real forms of the simple complex Lie algebras of types G_2 and F_4 were classified up to equivalence in [7]; some partial results were obtained for type E_6 in [8].

Here we will classify up to isomorphism all G-gradings on real forms of the classical simple complex Lie algebras except D_4. We do not use the topology of $\mathbb{R}$, so in all of our results $\mathbb{R}$ can be replaced by an arbitrary real closed field. We will follow the approach that has already been used over algebraically closed fields [2,12]: embed our Lie algebra $\mathcal{L}$ into an associative algebra $\mathcal{R}$ with involution φ and transfer the classification problem for gradings from $\mathcal{L}$ to $(\mathcal{R}, \varphi)$ using automorphism group schemes. Specifically:

- $\mathcal{R} = M_n(\mathbb{R})$, with φ orthogonal (that is, given by a symmetric bilinear form) and n odd, yield all real forms of series B by taking $\mathcal{L} = \mathrm{Skew}(\mathcal{R}, \varphi) := \{x \in \mathcal{R} \mid \varphi(x) = -x\}$;
- $\mathcal{R} = M_n(\mathbb{R})$, with φ symplectic (that is, given by a skew-symmetric bilinear form over $\mathbb{R}$, so n must be even), and $\mathcal{R} = M_n(\mathbb{H})$, with φ symplectic (that is, given by a hermitian form over $\mathbb{H}$), yield all real forms of series C by taking $\mathcal{L} = \mathrm{Skew}(\mathcal{R}, \varphi)$;
- $\mathcal{R} = M_n(\mathbb{R})$, with φ orthogonal (that is, given by a symmetric bilinear form over $\mathbb{R}$) and n even, and $\mathcal{R} = M_n(\mathbb{H})$, with φ orthogonal (that is, given by a skew-hermitian form over $\mathbb{H}$), yield all real forms of series D by taking $\mathcal{L} = \mathrm{Skew}(\mathcal{R}, \varphi)$;
- $\mathcal{R} = M_n(\mathbb{C})$, $M_n(\mathbb{R}) \times M_n(\mathbb{R})$ and $M_n(\mathbb{H}) \times M_n(\mathbb{H})$, with φ of the second kind (that is, nontrivial on the center of $\mathcal{R}$), yield all real forms of series A by taking $\mathcal{L} = \mathrm{Skew}(\mathcal{R}, \varphi)'$, where prime denotes the derived Lie algebra.