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Almost nilpotency of an associative algebra with an almost nilpotent fixed-point subalgebra

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ABSTRACT

Let A be an associative algebra of arbitrary dimension over a field F and G a finite group of automorphisms of A of order n, prime to the characteristic of F. Denote by $A^G = \{a \in A \mid a^g = a \text{ for all } g \in G\}$ the fixed-point subalgebra. By the classical Bergman–Isaacs theorem, if A^G is nilpotent of index d, i.e. $(A^G)^d = 0$, then A is also nilpotent and its nilpotency index is bounded by a function depending only on n and d. We prove, under the additional assumption of solubility of G, that if A^G contains a two-sided nilpotent ideal $I \lhd A^G$ of nilpotency index d and of finite codimension m in A^G, then A contains a nilpotent two-sided ideal $H \lhd A$ of nilpotency index bounded by a function of n and d and of finite codimension bounded by a function of m, n and d. An even stronger result is provided for graded associative algebras: if G is a finite (not necessarily soluble) group of order n and $A = \bigoplus_{g\in G} A_g$ is a G-graded associative algebra over a field F, i.e. $A_gA_h \subset A_{gh}$, such that the identity component A_e has a two-sided nilpotent ideal $I_e \lhd A^G$ of nilpotency index d and of finite codimension m in A_e, then A has a homogeneous nilpotent two-sided ideal $H \lhd A$ of nilpotency index bounded by a function of n and d and of finite codimension bounded by a function of n, d and m.

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1. Introduction

By the classical Bergman–Isaacs theorem [1], if an associative algebra A over a field F admits a finite group of automorphisms G of order $|G| = n$, prime to the characteristic of F, and the fixed-point subalgebra $A^G = \{a \in A \mid a^g = a \text{ for all } g \in G\}$ is nilpotent of index d, i.e. $(A^G)^d = 0$, then A is nilpotent of index bounded by a function of n and d. After this work, a great number of paper deal with properties of an algebra (or a ring) under a finite group action subject to some constraints on the fixed-point subalgebra. In this paper we prove, under the additional assumption of the solubility of the automorphism group, that the "almost nilpotency" of the fixed-point subalgebra implies the "almost nilpotency" of the algebra itself. Namely, the following theorem holds.

Theorem 1.1. *Let A be an associative algebra of arbitrary (possibly infinite) dimension over a field F acted on by a finite soluble group G of order n. Suppose that the characteristic of F does not divide n. If the fixed-point subalgebra A^G has a nilpotent two-sided ideal $I \lhd A^G$ of nilpotency index d and of finite codimension m in A^G, then A has a nilpotent two-sided ideal $H \lhd A$ of nilpotency index bounded by a function of n and d and of finite codimension bounded by a function of m, n and d.*

The restrictions on the order of the automorphism group are unavoidable. There are examples showing that the result is not true either for infinite automorphism groups or for algebras with n-torsion.

Theorem 1.1 follows by induction on the order of G from the Bergman–Isaacs theorem and the following statement on graded associative algebras, in which we do not suppose either G to be soluble or the order of G to be prime to the characteristic of the field.

Theorem 1.2. *Let G be a finite group of order n and let $A = \bigoplus_{g \in G} A_g$ be a G-graded associative algebra over a field F, i.e. $A_g A_h \subset A_{gh}$. If the identity component A_e has a nilpotent two-sided ideal $I_e \lhd A_e$ of nilpotency index d and of finite codimension m in A_e, then A has a homogeneous nilpotent two-sided ideal $H \lhd A$ of nilpotency index bounded by a function on n and d and of finite codimension bounded by a function on n, d and m.*

The proof of Theorem 1.2 is based on the method of generalized centralizers, originally created by Khukhro in [3] for nilpotent groups and Lie algebras with an almost regular automorphism of prime order. In [4,5] the approach was significantly revised and new techniques were introduced to study a more complicated case of an almost regular automorphism of arbitrary (not necessarily prime) finite order. In particular, it was proved that if a Lie algebra L admits an automorphism φ of finite order n with fixed-point subalgebra of finite dimension m, then L has a soluble ideal of derived length bounded by a function of n whose codimension is bounded by a function of m and n. The combinatorial nature of the construction in [5] makes possible to apply it to a wide range