Journal of Algebra 506 (2018) 188–214

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

# Hochschild cohomology and deformation quantization of affine toric varieties

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ARTICLE INFO

*Article history:*
Received 13 June 2017
Available online 22 March 2018
Communicated by Michel Van den Bergh

*MSC:*
13D03
13D10
14B05
14B07
14M25
53D55

*Keywords:*
Deformation quantization
Hochschild cohomology
Toric singularities

ABSTRACT

For an affine toric variety Spec$(A)$, we give a convex geometric description of the Hodge decomposition of its Hochschild cohomology. Under certain assumptions we compute the dimensions of the Hodge summands $T^1_{(i)}(A)$, generalizing the existing results about the André–Quillen cohomology group $T^1_{(1)}(A)$. We prove that every Poisson structure on a possibly singular affine toric variety can be quantized in the sense of deformation quantization.



## 1. Introduction

The concept of deformation quantization has been appearing in the literature for many years and was established by Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer in [5]. A major result, concerning the existence of deformation quantization is Kontsevich's formality theorem [20, Theorem 4.6.2] which implies that every Poisson structure on a real

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https://doi.org/10.1016/j.jalgebra.2018.03.016

manifold can be quantized, i.e., admits a star product. Kontsevich also extended the notion of deformation quantization into the algebro-geometric setting [19]. From Yekutieli's results [32], [33] it follows that on a smooth algebraic variety $X$ (under certain cohomological restrictions) every Poisson structure admits a star product. As in Kontsevich's case, the construction is canonical and induces a bijection between the set of formal Poisson structures modulo gauge equivalence and the set of star products modulo gauge equivalence (see also Van den Bergh's paper [31]).

When $X = \operatorname{Spec}(A)$ is a smooth affine variety, we have the following formality theorem: there exists an $L_\infty$-quasi-isomorphism between the Hochschild differential graded Lie algebra $C^\bullet(A)[1]$ and the formal differential graded Lie algebra $H^\bullet(A)[1]$ (i.e., the graded Lie algebra $H^\bullet(A)[1]$ with trivial differential), extending the Hochschild–Kostant–Rosenberg quasi-isomorphism of the above complexes. Dolgushev, Tamarkin and Tsygan [14] proved even a stronger statement by showing that the Hochschild complex $C^\bullet(A)$ is formal as a homotopy Gerstenhaber algebra. Consequently, every Poisson structure on a smooth affine variety can be quantized.

Studying non-commutative deformations (also called quantizations) of toric varieties is important for constructing and enumerating noncommutative instantons (see [9], [10]), which is closely related to the computation of Donaldson–Thomas invariants on toric threefolds (see [18], [13]).

In the paper we drop the smoothness assumption and consider the deformation quantization problem for possibly singular affine toric varieties. In the singular case the Hochschild–Kostant–Rosenberg map is no longer a quasi-isomorphism and thus also the $n$-th Hochschild cohomology group is no longer isomorphic to the Hodge summand $H^n_{(n)}(A) \cong \operatorname{Hom}_A(\Omega^n_{A|k}, A)$. Therefore, other components of the Hodge decomposition come into play, making the problem of deformation quantization interesting from the cohomological point of view. In general many parts of the Hodge decomposition are still unknown. The case of complete intersections has been settled in [15], where Frønsdal and Kontsevich also motivated the problem of deformation quantization on singular varieties. In the toric case Altmann and Sletsjøe [4] computed the Harrison parts of the Hodge decomposition.

Deformation quantization of singular Poisson algebras does not exist in general; see Mathieu [23] for counterexamples. For known results about quantizing singular Poisson algebras we refer the reader to [29] and references therein. The associative deformation theory for complex analytic spaces was developed by Palamodov in [25] and [26]. For recent developments concerning the problem of deformation quantization in derived geometry, see [8].

The paper is organized as follows: in Section 2 and 3 we recall definitions and some techniques for computing Hochschild cohomology. We compute the Hochschild cohomology of a reduced isolated hypersurface singularity in Proposition 3.3. Section 4 contains computations of Hochschild cohomology for toric varieties. In Theorem 4.9 we give a convex geometric description of the Hodge decomposition of the Hochschild cohomology for affine toric varieties. As an application we explicitly calculate $T^1_{(i)}(A)$ for all $i \in \mathbb{N}$