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Indecomposable decompositions of torsion-free abelian groups

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A B S T R A C T

An indecomposable decomposition of a torsion-free abelian group G of rank n is a decomposition $G = A_1 \oplus \cdots \oplus A_t$ where each A_i is indecomposable of rank r_i, so that $\sum_i r_i = n$ is a partition of n. The group G may have indecomposable decompositions that result in different partitions of n. We address the problem of characterizing those sets of partitions of n which can arise from indecomposable decompositions of a torsion-free abelian group.

1. Introduction

All the groups considered in this paper belong to the class $\mathcal{G}$ of finite rank torsion-free abelian groups; we shall refer to them simply as 'groups'. Any group $G \in \mathcal{G}$ is viewed

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as an additive subgroup of a finite dimensional $\mathbb{Q}$-vector space and its ***rank***, $\mathrm{rk}(G)$, is the dimension of the subspace $\mathbb{Q}G$ that G generates. If a group G of rank n has an indecomposable decomposition $G = A_1 \oplus \cdots \oplus A_t$ where A_i is indecomposable of rank r_i, then $\sum_i r_i = n$ so $P = (r_1, \ldots, r_t)$ is an unordered partition of n into t parts, and we say that G ***realizes*** P.

A group G may realize several different partitions and we denote by $\mathcal{P}(G)$ the set of all partitions that are realized by G. Determining $\mathcal{P}(G)$ for a given group G and characterizing realizable sets of partitions have received a great deal of attention in the literature. For example, [6, Section 90] displays groups realizing $(1, 3)$ and $(2, 2)$, groups realizing partitions into 2 and t parts for each $2 \leq t < n$, and groups realizing every partition of n into t parts (Corner's Theorem 2.1 below).

On the other hand, there are classes of groups, such as completely decomposable groups, direct sums of indecomposable fully invariant subgroups, and almost completely decomposable groups with primary regulating index (see [5, Theorem 3.5]) that realize a unique partition. Furthermore, some sets of partitions are incompatible, in the sense that no group can realize them. For example, we showed in [10, Theorem 3.5] (2.3) that no group can realize both $(k, 1^{n-k})$ and $(m, 1^{n-m})$ if $k \neq m$, where $(i, 1^j)$ stands for the partition $(i, 1, \ldots, 1)$ with j copies of 1. In this paper, we consider three problems:

1. Characterize families of partitions that can be realized by some group.
2. For a given family $\mathcal{P}$ of partitions of n that can be realized, determine the groups G that realize $\mathcal{P}$.
3. For a given class $\mathcal{A}$ of groups, determine the families $\mathcal{P}(G)$ of partitions realized by $G \in \mathcal{A}$.

For the most part, we adopt the notation of [6]. The definitions and properties of almost completely decomposable, rigid and block-rigid cyclic regulator quotient groups can be found in [9]. The properties we need are described in Section 2.

This paper contains the following results. By $\mathcal{B}$ we denote the class of all block-rigid almost completely decomposable groups with cyclic regulator quotient.

1. The "Hook condition" is shown to be necessary for the sets of partitions $\mathcal{P}(X)$ of groups $X \in \mathcal{B}$ (3.1(1)).
2. A useful "standard" description of $\mathcal{B}$-groups is established (5.12).
3. The workhorse Lemma 6.7 describes the direct sum of two $\mathcal{B}$-groups given in standard description.
4. General obstructions to realization are established (Lemma 4.2, Lemma 4.3, Lemma 6.1).
5. The maximal sets of partitions of n that satisfy the Hook condition are described and denoted by $\mathcal{S}(n, k)$ (Proposition 3.3). The (few) $\mathcal{S}(n, k)$ that can be realized by groups in $\mathcal{B}$ and the (many) that cannot so realized are determined (Theorem 7.2).