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On the coefficients of the Alekseev–Torossian associator

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ABSTRACT

This paper explains a method to calculate the coefficients of the Alekseev–Torossian associator as linear combinations of iterated integrals of Kontsevich weight forms of Lie graphs.

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0. Introduction

Associators are group-like non-commutative formal power series with two variables which were subject to the pentagon equation and the hexagon equations in [9] (actually it was shown in [13] that the former implies the latter). The notion is involved with wide area of mathematics, the quantization of Lie-bialgebras (cf. [10]), the combinatorial reconstruction of the universal Vassiliev knot invariant (cf. [5,6,15,19,21]), the proof of formality of chain operad of little discs (cf. [24,23]), the solution of Kashiwara–Vergne conjecture (cf. [3]), etc.

A typical example of associators is the *KZ-associator* Φ_{KZ} in the algebra $\mathbb{C}\langle\langle A, B\rangle\rangle$ of power series over $\mathbb{C}$ with variables A and B, which was constructed by two fundamental solutions of the KZ (Knizhnik–Zamolodchikov) equation in [9]. In [20] Theorem A.8 and [12] Proposition 3.2.3, it was given a method to calculate its coefficients as linear combinations of *multiple zeta values*, the real numbers defined by the following power series

$$\zeta(k_1, \ldots, k_m) := \sum_{0<n_1<\cdots<n_m} \frac{1}{n_1^{k_1}\cdots n_m^{k_m}}$$

with $k_1, \ldots, k_m \in \mathbb{N}$ and $k_m > 1$ (the condition to be convergent).

The AT-associator Φ_{AT} is another example of associators. It was introduced by Alekseev and Torossian [2] as an 'associator in TAut_3' and later shown to be an associator in $\mathbb{R}\langle\langle A, B\rangle\rangle$ by Ševera and Willwacher [23]. It was constructed by a parallel transport of the AT-equation (cf. §3) on Kontsevich's eye $\overline{C}_{2,0}$ (cf. §1). This paper discusses a AT-counterpart of the results of [20] and [12]. We give a method in Theorem 3.3 to describe coefficients of the AT-associator Φ_{AT} in terms of linear combinations of iterated integrals of Kontsevich weight forms of Lie graphs (cf. §2) on $\overline{C}_{2,0}$ and execute computations in lower depth in §4.

We note that similar (or possibly related) arguments are observed in [4] Theorem 8.0.4.5 where Alm described Lyndon word expansion of the 1-form ω_{AT}, while in this paper we calculate free word expansion of its parallel transport Φ_{AT}.

1. Kontsevich's eye

We will recall the compactified configuration spaces [16]. Let $n \geqslant 1$. For a topological space X, we define $\mathrm{Conf}_n(X) := \{(x_1, \ldots, x_n) \mid x_i \neq x_j \ (i \neq j)\}$. The group $\mathrm{Aff}_+ := \{x \mapsto ax + b \mid a \in \mathbb{R}_+^\times, b \in \mathbb{C}\}$ acts on $\mathrm{Conf}_n(\mathbb{C})$ diagonally by rescallings and parallel translations. We denote the quotient by

$$C_n := \mathrm{Conf}_n(\mathbb{C})/\mathrm{Aff}_+$$

for $n \geqslant 2$, which is a connected oriented smooth manifold with dimension $2n - 3$. E.g. $C_2 \simeq S^1$ and $C_3 \simeq S^1 \times (\mathrm{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\})$.