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On the comparison of two constructions of Witt vectors of non-commutative rings

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ABSTRACT

Let A be any associative ring, possibly non-commutative and let p be a prime number. Let $E(A)$ be the ring of p-typical Witt vectors as constructed by Cuntz and Deninger in [1] and $W(A)$ be that constructed by Hesselholt in [3]. The goal of this paper is to answer the following question by Hesselholt: Is $HH_0(E(A)) \cong W(A)$? We show that in the case $p = 2$, there is no such isomorphism possible if one insists that it be compatible with the Verschiebung operator and the Teichmüller map.

1. Introduction

Let A be a associative unital ring and p be a prime number. When A is commutative, the classical construction of p-typical Witt vectors gives us a topological ring $W(A)$, equipped with a Verschiebung operator

$$W(A) \xrightarrow{V} W(A)$$

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and a Teichmüller map, which we denote by

$$A \xrightarrow{\langle\ \rangle} W(A).$$

In this paper we consider two generalizations of this construction to the non-commutative case. One of them is a construction of an abelian group $W(A)$ given by Hesselholt in [2] (see also [3]). The other is a construction of a ring $E(A)$ by Cuntz and Deninger, given in [1]. Just as in the commutative case, $E(A)$ and $W(A)$ are topological rings and are equipped with the Verschiebung operator and the Teichmüller map. Moreover, both $W(A)$ and $E(A)$ are isomorphic to the classical construction of Witt vectors when A is commutative. Let $HH_0(E(A)) := E(A)/\overline{[E(A), E(A)]}$. The goal of this paper is to answer the following question of Hesselholt.

Question 1.1. Is $W(A)$ isomorphic to $HH_0(E(A))$?

We note that $HH_0(E(A))$ inherits the Verschiebung operator V and the Teichmüller map $\langle\ \rangle$, from $E(A)$. In this paper we only consider maps from $W(A)$ to $HH_0(E(A))$ which are compatible with V and $\langle\ \rangle$.

The following is one of the main results of this paper.

Theorem 1.2. *Let $A = \mathbb{Z}\{X, Y\}$ and $p = 2$. Then*

(i) *$W(A)$ is topologically generated by $\{V^n(\langle a\rangle) \mid n \in \mathbb{N}_0, a \in A\}$.*
(ii) *$HH_0(E(A))$ is not topologically generated by $\{V^n(\langle a\rangle) \mid n \in \mathbb{N}_0, a \in A\}$.*
(iii) *there is no continuous surjective map from $W(A) \to HH_0(E(A))$ which commutes with V and is compatible with $\langle\ \rangle$.*

One can thus slightly modify Question 1.1 and ask for existence of a map (not necessarily surjective) from $W(A) \to HH_0(E(A))$ compatible with the Verschiebung operator and the Teichmüller map. The following theorem, shows that even this is not possible, at least in the case $p = 2$.

Theorem 1.3. *Let $A := \mathbb{Z}\{X, Y\}$ and $p = 2$. Then there is no continuous group homomorphism from $W(A) \to HH_0(E(A))$ which is compatible with V and $\langle\ \rangle$.*

We believe that Theorems 1.2 and 1.3 should hold for all primes p.

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