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On certain tilting modules for SL_2

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ABSTRACT

We give a complete picture of when the tensor product of an induced module and a Weyl module is a tilting module for the algebraic group SL_2 over an algebraically closed field of characteristic p. Whilst the result is recursive by nature, we give an explicit statement in terms of the p-adic expansions of the highest weight of each module.

1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$, and let G be the group $SL_2(k)$. In this article we investigate the tensor product $\nabla(r) \otimes \Delta(s)$ of the induced module of highest weight r and the Weyl module of highest weight s. Similar tensor products for $SL_2(k)$ have been studied before, in particular the product $L(r) \otimes L(s)$ of corresponding simple modules, by Doty and Henke in 2005 [3]. Motivated by their results utilising tilting modules, we describe exactly when the product $\nabla(r) \otimes \Delta(s)$ is a tilting module.

By an argument of Donkin given in [6, Lemma 3.3], it's known already that when $|r - s| \leq 1$ the module $\nabla(r) \otimes \Delta(s)$ is tilting. Some other special cases are also known,

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for example the tensor product of Steinberg modules $\nabla(p^n - 1) \otimes \Delta(p^m - 1)$ is tilting, since $\nabla(p^k - 1) = \Delta(p^k - 1)$ for all $k \in \mathbb{N}$, as is the tensor product $\nabla(a) \otimes \Delta(b)$ for $a, b \in \{0, \dots, p-1\}$.

Before stating the main theorem of this paper, we introduce some notation. Let $r \in \mathbb{N}$ and p a prime. We write the base p expansion

$$r = \sum_{i=0}^{n} r_i p^i,$$

where each $r_i \in \{0, \dots, p-1\}$, $r_n \neq 0$ and for all $j > n$ we set $r_j = 0$. We say that r has p-length n (or just length n if the prime is clear), and write $\mathrm{len}_p(r) = n$. We define $\mathrm{len}_p(0) = -1$. Now given any pair $(r, s) \in \mathbb{N}^2$ we can write

$$r = \sum_{i=0}^{n} r_i p^i, \quad s = \sum_{i=0}^{n} s_i p^i$$

where $n = \max(\mathrm{len}_p(r), \mathrm{len}_p(s))$ so that at least one of r_n and s_n is non zero. If $r \neq s$, let m be the largest integer such that $r_m \neq s_m$ and let

$$\hat{r} = \sum_{i=0}^{m} r_i p^i, \quad \hat{s} = \sum_{i=0}^{m} s_i p^i$$

so that if $r > s$ we have $r_m > s_m$ and $\hat{r} > \hat{s}$. In the case $r = s$, we define $\hat{r} = \hat{s} = 0$. We call the pair $(\hat{r}, \hat{s})$ the primitive of (r, s), and say that (r, s) is a primitive pair if $(r, s) = (\hat{r}, \hat{s})$.

Theorem 1.1. *Let the pair $(\hat{r}, \hat{s})$ be the primitive of (r, s). The module $\nabla(r) \otimes \Delta(s)$ is a tilting module if and only if one of the following*

1. $\hat{r} = p^n - 1 + ap^n$ *for some* $a \in \{0, \dots, p-2\}$, $n \in \mathbb{N}$, *and* $\hat{s} < p^{n+1}$,
2. $\hat{s} = p^n - 1 + bp^n$ *for some* $b \in \{0, \dots, p-2\}$, $n \in \mathbb{N}$, *and* $\hat{r} < p^{n+1}$.

Fig. 1 illustrates which of the modules $\nabla(r) \otimes \Delta(s)$ are tilting for $r, s \leq 31$ and $p = 2$.

1.1. Terminology

In this section, we fix some terminology. Throughout, k will be an algebraically closed field of characteristic $p > 0$, and G will be the affine algebraic group $SL_2(k)$. Let B be the Borel subgroup of G consisting of lower triangular matrices and containing the maximal torus T of diagonal matrices. Let $X(T)$ be the weight lattice, which we associate with $\mathbb{Z}$ in the usual manner. Under this association the set of dominant weights X^+ corresponds to the set $\mathbb{N} \cup \{0\}$.