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Brauer relations for finite groups in the ring of semisimplified modular representations

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A B S T R A C T

Let G be a finite group and p be a prime. We study the kernel of the map, between the Burnside ring of G and the Grothendieck ring of $\mathbb{F}_p[G]$-modules, taking a G-set to its associated permutation module. We are able, for all finite groups, to classify the primitive quotient of the kernel; that is for each G, the kernel modulo elements coming from the kernel for proper subquotients of G. We are able to identify exactly which groups have non-trivial primitive quotient and we give generators for the primitive quotient in the soluble case.

1. Introduction

In this paper we will describe, for a prime p and a finite group G, the kernel of the map $m_{\mathbb{F}_p,ss}(G)$ from the Burnside ring $b(G)$ to the Grothendieck ring of $\mathbb{F}_p[G]$ modules $G_0(\mathbb{F}_p[G])$ (equivalently the ring of Brauer characters), which takes the isomorphism class of a finite G-set X to the class of $\mathbb{F}_p[X]$ in $G_0(\mathbb{F}_p[G])$. The $\mathbb{F}_p[G]$-module $\mathbb{F}_p[X]$ has an $\mathbb{F}_p$-basis indexed by elements of X, G acts by permuting this basis in the obvious

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way. The class of $\mathbb{F}_p[X]$ in $G_0(\mathbb{F}_p[G])$ is equal to that of its semisimplification and is by definition determined by its composition factors. Note that composition factors do not in general determine isomorphism class, so two isomorphism classes of G-sets may have the same image in $G_0(\mathbb{F}_p[G])$ even if their associated permutation modules are not isomorphic.

Definition 1.1. Let G be a finite group. The additive group of the *Burnside ring* $b(G)$ is the free abelian group on isomorphism classes of finite G sets modulo the relations $[X] + [Y] - [X \coprod Y]$, where $[X]$ denotes the isomorphism class of X. Furthermore, $b(G)$ is equipped with a ring structure defined by $[X] \cdot [Y] = [X \times Y]$.

Remark 1.2. The Burnside ring is isomorphic to the free abelian group on isomorphism classes of transitive G sets. Moreover, by the orbit-stabiliser theorem each transitive G-set is isomorphic to G/H, where $H \leqslant G$ is a stabiliser of an element of the G-set, and is determined uniquely up to G-conjugacy. Thus we may identify $b(G)$, as an abelian group, with the free abelian group on conjugacy classes of subgroups of G.

Definition 1.3. Let G be a finite group and R be a commutative ring, the *Grothendieck Group* of $R[G]$ modules $G_0(R[G])$ is defined to be the free abelian group on isomorphism classes of $R[G]$-modules modulo the relations $[A]-[C]+[B]$ for every short exact sequence of $R[G]$-modules:

$$0 \to A \to C \to B \to 0.$$

Furthermore, there is a ring structure with multiplication given by $[A] \cdot [B] = [A \otimes_R B]$.

Consider the map:

$$\begin{aligned} m_{\mathbb{F}_p,ss}(G) : b(G) &\to G_0(\mathbb{F}_p[G]) \\ [H] &\mapsto [\mathrm{Ind}_{G/H}(1)], \end{aligned}$$

where $\mathrm{Ind}_{G/H}(-)$ denotes the induction map from H to G and 1 denotes the trivial $\mathbb{F}_p[H]$-module. Elements of the kernel $K_{\mathbb{F}_p,ss}(G)$ will be called *Brauer Relations* for G over $\mathbb{F}_p$ semisimplified or relations over $\mathbb{F}_p, ss$ for short.

Following [1] and [2] a relation for G is said to be imprimitive if it is a linear combination of relations which are "inflated" from proper quotients of G or "induced" from proper subgroups. We will then describe, for a finite group G and prime p, the structure of the kernel modulo imprimitive relations as an abelian group and describe explicit generators in the case G is soluble. More precisely our main aim is the following theorem:

Theorem 1.4. *Let p be a prime, and G be a finite group of order divisible by p, all elements of* $\ker(m_{\mathbb{F}_p,ss}(G))$ *are linear combinations of relations "induced" or "inflated" from* $\ker(m_{\mathbb{F}_p,ss}(H))$ *from subquotients H of the following forms:*