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Towards functor exponentiation

Mikhail Khovanov [a], Yin Tian [b,*]

[a] *Department of Mathematics, Columbia University, New York, NY 10027, United States*
[b] *Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China*

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ABSTRACT

We consider a possible framework to categorify the exponential map $\exp(-f)$ given the categorification of a generator f of $\mathfrak{sl}_2$ by Lauda. In this setup the Taylor expansions of $\exp(-f)$ and $\exp(f)$ turn into complexes built out of categorified divided powers of f. Hom spaces between tensor powers of categorified f are given by diagrammatics combining nilHecke algebra relations with those for a additional "short strand" generator. The proposed framework is only an approximation to categorification of exponentiation, because the functors categorifying $\exp(f)$ and $\exp(-f)$ are not invertible.

1. Introduction

The exponential function is fundamental in mathematics. In Lie theory, the exponential map connects a Lie algebra and its Lie group. Idempotented version of quantized universal enveloping algebras of simple Lie algebras have been categorified [3,4,8]. This

* Corresponding author.
E-mail addresses: khovanov@math.columbia.edu (M. Khovanov), ytian@math.tsinghua.edu.cn (Y. Tian).

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paper can be viewed as a small step towards lifting the exponential map to the categorical level.

We focus on the case of $\mathfrak{sl}_2$. Consider the expansion

$$\exp(-f) = \sum_{k\geq 0}(-1)^k \frac{f^k}{k!},$$

in a completion of the universal enveloping algebra of $\mathfrak{sl}_2$, where $f \in \mathfrak{sl}_2$ is a Chevalley generator of the lower-triangular matrix. Categorification of the divided power $f^{(k)} = \frac{f^k}{k!}$ and of its quantized version $\frac{f^k}{[k]!}$ naturally appears in the categorified quantum $\mathfrak{sl}_2$ [7]. The generator f is lifted to a bimodule $\mathcal{F}'$ over a direct sum of the nilHecke algebras $\bigoplus_{n\geq 0} NH_n$. The tensor powers $\mathcal{F}'^{\,k} = \mathcal{F}'^{\otimes k}$ of the bimodule admit a direct sum decomposition $\mathcal{F}'^{\,k} \cong \bigoplus_{k!} \mathcal{F}'^{\,(k)}$. It is natural to expect lifting $\exp(-f)$ to a cochain complex whose degree k component is $\mathcal{F}'^{\,(k)}$ for $k \geq 0$. A nontrivial differential is needed to link adjacent components.

The diagrammatic approach is widely used in categorification and plays a significant role in the present paper as well. We provide a modification $\widetilde{NH}$ of $\bigoplus_{n\geq 0} NH_n$ by adding an extra generator to the nilHecke algebras The new generator is described by a short strand which links NH_n and NH_{n+1} together. The induction $\widetilde{NH}$-bimodule still exists, denoted $\mathcal{F}$. Short strand induces a $\widetilde{NH}$-bimodule homomorphism $\widetilde{NH} \to \mathcal{F}$. This morphism and its suitable generalizations $\mathcal{F}^k \to \mathcal{F}^{k+1}$ define a differential on $\bigoplus_{k\geq 0} \mathcal{F}^{(k)}[-k]$. The resulting complex descends to an alternating sum $\sum_{k\geq 0}(-1)^k[\mathcal{F}^{(k)}]$ in the Grothendieck ring of the derived category of $\widetilde{NH}$-bimodules.

Due to the existence of the short strand, the extension group $\mathrm{Ext}^1(\mathcal{F}, \widetilde{NH})$ of bimodules is nontrivial. We lift the expansion $\exp(f) = \sum_{k\geq 0} f^{(k)}$ to a complex $\left(\bigoplus_{k\geq 0} \mathcal{F}^{(k)}, d\right)$, where the differential d consists of certain elements of Ext^1-groups. The absence of the alternating sign in the expansion of $\exp(f)$ comes from the fact that it cancels against the sign coming from the use of Ext^1-groups.

Unfortunately, the two resulting complexes, lifting $\exp(-f)$ and $\exp(f)$, respectively, are not invertible, as explained in Section 4. A more elaborate or just a different construction is needed to more adequately categorify exponentiation.

Problem 1.1. Find a framework for categorification of the exponential map, where an object $\mathcal{F}$ in a monoidal triangulated category $\mathcal{C}$ lifts to two invertible objects $\exp(\mathcal{F})$ and $\exp(-\mathcal{F})$ in some monoidal triangulated category $\mathcal{C}^e$. The objects should descend to $\exp([\mathcal{F}])$ and $\exp(-[\mathcal{F}])$ in the Grothendieck ring of $\mathcal{C}^e$, where $[\mathcal{F}]$ is the class of $\mathcal{F}$ in the Grothendieck ring of $\mathcal{C}$. The Grothendieck rings of $\mathcal{C}$ and $\mathcal{C}^e$ should be suitably related.