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Action of automorphisms on irreducible characters of symplectic groups

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ABSTRACT

Assume G is a finite symplectic group $\mathrm{Sp}_{2n}(q)$ over a finite field $\mathbb{F}_q$ of odd characteristic. We describe the action of the automorphism group $\mathrm{Aut}(G)$ on the set $\mathrm{Irr}(G)$ of ordinary irreducible characters of G. This description relies on the equivariance of Deligne–Lusztig induction with respect to automorphisms. We state a version of this equivariance which gives a precise way to compute the automorphism on the corresponding Levi subgroup; this may be of independent interest. As an application we prove that the global condition in Späth's criterion for the inductive McKay condition holds for the irreducible characters of $\mathrm{Sp}_{2n}(q)$.

1. Introduction

1.1. The representation theory of finite groups is abound with many deep and fascinating conjectures nicknamed local/global conjectures; the paradigm of these is the McKay Conjecture. In a landmark paper [16] Isaacs–Malle–Navarro showed that the

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McKay Conjecture holds for all finite groups if a list of (stronger) conditions, jointly referred to as the inductive McKay condition, holds for the universal covering group of each finite simple group. We note that in the wake of [16] several other major local/global conjectures have also been reduced to checking certain inductive conditions.

1.2. Showing that these inductive conditions hold has revealed itself to be a difficult problem. One of the main difficulties arises from the fact that one needs some knowledge of how the automorphism group $\mathrm{Aut}(G)$, of a quasisimple group G, acts on the set of irreducible characters $\mathrm{Irr}(G)$. When the simple quotient $G/Z(G)$ is a group of Lie type then this question has turned out to be surprisingly vexing considering the large amount of machinery at our disposal. The main result of this paper gives a complete solution to this problem when $G = \mathrm{Sp}_{2n}(q)$ and q is a power of an odd prime. Note that the corresponding statement when q is even is an easy consequence of known results from Lusztig's classification of irreducible characters [20], see also [5] and [3].

1.3. In [30, Theorem 2.12] Späth gave a version of the inductive McKay condition which is specifically tailored to the finite groups of Lie type. There are several conditions in this statement, some of which are global and some of which are local. As an application of our result we show that the global condition concerning the stabilisers of irreducible characters of G under automorphisms holds for finite symplectic groups over fields of odd characteristic, see Theorem 16.2. We note that Cabanes and Späth [4] have recently shown the whole inductive McKay condition holds for the symplectic groups, hence independently proving Theorem 16.2 using completely different methods.

1.4. To state our result we need to introduce some notation. Let $\mathbf{G}$ be a connected reductive algebraic group defined over an algebraic closure $\overline{\mathbb{F}}_p$ of the finite field $\mathbb{F}_p$ of prime order p. We assume $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius endomorphism endowing $\mathbf{G}$ with an $\mathbb{F}_q$-rational structure $\mathbf{G}^F$. If $\mathbf{G}^{\star F^\star}$ is a group dual to $\mathbf{G}^F$ then to each semisimple element $s \in \mathbf{G}^{\star F^\star}$ we have a corresponding rational Lusztig series $\mathcal{E}(\mathbf{G}^F, s) \subseteq \mathrm{Irr}(\mathbf{G}^F)$ of irreducible characters. Moreover, we have $\mathrm{Irr}(\mathbf{G}^F) = \bigsqcup_{[s]} \mathcal{E}(\mathbf{G}^F, s)$ is a disjoint union of these series where the union runs over all the $\mathbf{G}^{\star F^\star}$-conjugacy classes of semisimple elements. As we will see, a particularly important role is played by those series $\mathcal{E}(\mathbf{G}^F, s)$ for which s is a quasi-isolated in $\mathbf{G}^\star$. We recall that this means $C_{\mathbf{G}^\star}(s)$ is not contained in any proper Levi subgroup of $\mathbf{G}^\star$.

1.5. If $\mathrm{Aut}(\mathbf{G}^F)$ denotes the automorphism group of $\mathbf{G}^F$ then we set ${}^\sigma\chi = \chi \circ \sigma^{-1}$ for any $\sigma \in \mathrm{Aut}(\mathbf{G}^F)$ and $\chi \in \mathrm{Irr}(\mathbf{G}^F)$; this defines an action of $\mathrm{Aut}(\mathbf{G}^F)$ on $\mathrm{Irr}(\mathbf{G}^F)$. The automorphism group $\mathrm{Aut}(\mathbf{G}^F)$ is well known to be generated by inner, diagonal, field, and graph automorphisms. In the case of diagonal automorphisms the action on $\mathrm{Irr}(\mathbf{G}^F)$ is well understood by work of Lusztig [23]. Moreover, in the case of symplectic groups defined over a field of odd characteristic there are no graph automorphisms. With this in place we may state our main result.

Theorem 1.6. *Assume $\mathbf{G}^F = \mathrm{Sp}_{2n}(q)$ with q odd. If $\sigma \in \mathrm{Aut}(\mathbf{G}^F)$ is a field automorphism and $\chi \in \mathcal{E}(\mathbf{G}^F, s)$ is an irreducible character, with $s \in \mathbf{G}^{\star F^\star}$ quasi-isolated in $\mathbf{G}^\star$, then we have ${}^\sigma\chi = \chi$.*