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New examples of dimension zero categories

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A B S T R A C T

We say that a category $\mathscr{D}$ is dimension zero over a field F provided that every finitely generated representation of $\mathscr{D}$ over F is finite length. We show that Rel(R), a category that arises naturally from a finite idempotent semiring R, is dimension zero over any infinite field. One special case of this result is that Rel, the category of finite sets with relations, is dimension zero over any infinite field.

1. Introduction and preliminaries

We define a representation of a category $\mathscr{D}$ over a field F to be a functor from $\mathscr{D}$ to $Vect_F$, the category of vector spaces over F. We say that a category $\mathscr{D}$ is dimension zero over a field F provided that every finitely generated representation of $\mathscr{D}$ over F is finite length. The purpose of this paper is to show that Rel, the category of finite sets with relations, is dimension zero over any infinite field. Our method of argument allows this result to be generalized to categories that we call Rel(R), where R is any finite idempotent semiring (Definition 1.8). Bouc and Thévenaz [1] have independently shown that Rel, the category of finite sets with relations, is dimension zero over any field. Theorem 3.2 states that Rel(R) is dimension zero over any infinite field for any finite idempotent semiring R.

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For the rest of this paper, let $\mathscr{D}$ be a combinatorial category, i.e. a category such that $\mathrm{Hom}(a,b)$ is finite for all objects $a,b \in \mathscr{D}$, and let F be a field. We will let $Vect_F$ denote the category of vector spaces over F; the objects are vector spaces over F and the morphisms are linear transformations. Finally, for the rest of this paper, let $[n]$ be the set $\{1,...,n\}$ for any whole number n.

We will now introduce several notions in representation theory which will be important in this paper.

Definition 1.1. A representation of $\mathscr{D}$ over F is a functor from $\mathscr{D}$ to $Vect_F$.

Concretely, a representation V of $\mathscr{D}$ over F takes every object $d \in \mathscr{D}$ to a vector space $V(d)$ over F, takes every morphism $g \in \mathrm{Hom}(d,e)$ to a linear map $V(g) \in \mathrm{Hom}(V(d),V(e))$ for all $d,e \in \mathscr{D}$, and satisfies the following two properties.

- $V(f \circ g) = V(f) \circ V(g)$ for all $f \in \mathrm{Hom}(y,z), g \in \mathrm{Hom}(x,y)$ for all $x,y,z \in \mathscr{D}$
- $V(Id_d) = Id_{V(d)}$ for all $d \in \mathscr{D}$

Definition 1.2. Let V be a representation of $\mathscr{D}$ over F. A subrepresentation of V is a representation W of $\mathscr{D}$ over F such that $W(d)$ is a vector subspace of $V(d)$ for all $d \in \mathscr{D}$ and $W(f)$ is the restriction of $V(f)$ to $W(d)$ for all $d,d' \in \mathscr{D}$ and $f \in \mathrm{Hom}_{\mathscr{D}}(d,d')$.

Two particularly easy examples of a representation of $\mathscr{D}$ over F are the zero representation and the trivial representation. The zero representation sends every object of $\mathscr{D}$ to 0 and every morphism in $\mathscr{D}$ to the zero transformation. The trivial representation sends every object of $\mathscr{D}$ to F and every morphism in $\mathscr{D}$ to the identity transformation.

Definition 1.3. A representation V is irreducible provided that V is not the zero representation and that the only subrepresentations of V are the zero representation and V itself.

Definition 1.4. A representation V of $\mathscr{D}$ over F is finitely generated provided that there exist objects $d_1,...,d_i \in \mathscr{D}$ and $v_{1,1},...,v_{1,j_1} \in V(d_1),...,v_{i,1},...,v_{i,j_i} \in V(d_i)$ such that if W is a subrepresentation of V and $v_{1,1},...,v_{1,j_1} \in W(d_1),...,v_{i,1},...,v_{i,j_i} \in W(d_i)$ then $W = V$.

Definition 1.5. A representation V is finite length provided that any non-repetitive chain of subrepresentations of V is finite.

An equivalent definition of finite length is that a representation V is finite length provided that there exists some non-repetitive finite chain $0 = W_0 \subsetneq ... \subsetneq W_n = V$ of subrepresentations of V such that each W_{i+1}/W_i is irreducible. When this is the case, $W_0,...,W_n$ is called a composition series for V and n is called the length of V. The Jordan–Hölder Theorem guarantees that if $W'_0,...,W'_m$ is another composition series for