Journal of Algebra 505 (2018) 279–287

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Dual Ore's theorem on distributive intervals of finite groups

Sebastien Palcoux

Institute of Mathematical Sciences, Chennai, India

ARTICLE INFO

Article history:
Received 10 August 2017
Available online 22 March 2018
Communicated by Martin Liebeck

MSC:
20D30
05E15
20C15
06C15

Keywords:
Finite group
Complex representation
Subgroup lattice
Distributive
Boolean

ABSTRACT

This paper gives a self-contained group-theoretic proof of a dual version of a theorem of Ore on distributive intervals of finite groups. We deduce a bridge between combinatorics and representations in finite group theory.

1. Introduction

Øystein Ore proved in 1938 that a finite group is cyclic if and only if its subgroup lattice is distributive, and he extended one side as follows, where $[H, G]$ will be an interval in the subgroup lattice of the group G (idem throughout the paper).

E-mail address: sebastienpalcoux@gmail.com.

https://doi.org/10.1016/j.jalgebra.2018.03.017
0021-8693/

Theorem 1.1 *([4]). Let $[H,G]$ be a distributive interval of finite groups. Then there is $g \in G$ such that $\langle Hg \rangle = G$.*

This paper first recalls our short proof of Theorem 1.1 and then gives a self-contained group-theoretic proof of the following dual version, where $G_{(V^H)}$ will be the pointwise stabilizer subgroup of G for the fixed-point subspace V^H (see Definition 3.1).

Theorem 1.2. *Let $[H,G]$ be a distributive interval of finite groups. Then there exists an irreducible complex representation V of G such that $G_{(V^H)} = H$.*

We deduce a bridge between combinatorics and representations:

Corollary 1.3. *The minimal number of irreducible components for a faithful complex representation of a finite group G is at most the minimal length ℓ for an ordered chain of subgroups*

$$\{e\} = H_0 < H_1 < \cdots < H_\ell = G$$

such that $[H_i, H_{i+1}]$ is distributive (or better, bottom Boolean).

It is a non-trivial upper bound involving the subgroup lattice only. These results were first proved by the author as applications to finite group theory of results on planar algebras [5, Corollaries 6.10, 6.11]. For the convenience of the reader and for being self-contained, this paper reproduces some preliminaries of [1] and [5].

2. Ore's theorem on distributive intervals

2.1. Basics in lattice theory

We refer to [7] for the notions of *finite lattice* L, *meet* $\wedge$, *join* $\vee$, *subgroup lattice* $\mathcal{L}(G)$, *sublattice* $L' \subseteq L$, *interval* $[a,b] \subseteq L$, *minimum* $\hat{0}$, *maximum* $\hat{1}$, *atom*, *coatom*, *distributive lattice*, *Boolean lattice* $\mathcal{B}_n$ (of rank n) and *complement* $b^{\complement}$ (with $b \in \mathcal{B}_n$). The *top interval* of a finite lattice L is the interval $[t, \hat{1}]$, with t the meet of all the coatoms. The *bottom interval* of a finite lattice L is the interval $[\hat{0}, b]$, with b the join of all the atoms. A lattice with a Boolean top interval will be called *top Boolean*; idem for *bottom Boolean.*

Lemma 2.1. *A finite distributive lattice is top and bottom Boolean.*

Proof. See [7, items a-i p254-255] which uses Birkhoff's representation theorem (a finite lattice is distributive if and only if it embeds into some $\mathcal{B}_n$). $\square$