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Euler characteristic of analogues of a Deligne–Lusztig variety for GL_n

Dongkwan Kim

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, USA

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A B S T R A C T

We give a combinatorial formula to calculate the Euler characteristic of an analogue of a Deligne–Lusztig variety, denoted $\mathcal{Y}_{w,g}$, which is attached to an element w in the Weyl group of GL_n and $g \in GL_n$. The main theorem of this paper states that the Euler characteristic of $\mathcal{Y}_{w,g}$ only depends on the unipotent part of the Jordan decomposition of g and the conjugacy class of w. It generalizes the formula of the Euler characteristic of Springer fibers for type A.

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E-mail address: sylvaner@math.mit.edu.

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1. Introduction

Suppose that an algebraic reductive group G over an algebraically closed field $\mathbf{k}$ is given. When $\mathbf{k}$ is an algebraic closure of a finite field with some fixed Frobenius morphism, Deligne and Lusztig [3] defined $X(w)$ for any element w in the Weyl group of G which is now called a Deligne–Lusztig variety. Likewise, we define $\mathcal{Y}_{w,g}$ to be the subvariety of the flag variety of G, consisting of Borel subgroups $B \subset G$ such that B and gBg^{-1} are in relative position w. Equivalently, $\mathcal{Y}_{w,g}$ is obtained from $X(w)$ by replacing the Frobenius morphism with conjugation by $g \in G$. The variety $\mathcal{Y}_{w,g}$ is studied by e.g. [9], [12], [15], [17], [18], [10], etc. Also when w is the identity, it coincides with the definition of the Springer fiber corresponding to $g \in G$.

The main result of this paper asserts that the Euler characteristic of $\mathcal{Y}_{w,g}$, denoted by $\chi(\mathcal{Y}_{w,g})$, is easy to calculate for $G = GL_n(\mathbf{k})$. Indeed, it only depends on the unipotent part of g in its Jordan decomposition and the conjugacy class of w in the Weyl group of G. Also there is a simple combinatorial formula to calculate such $\chi(\mathcal{Y}_{w,g})$. This generalizes the well-known formula of the Euler characteristic of Springer fibers for type A, cf. [20], [14], [6]. We expect that similar properties hold for reductive groups of other types.

2. Some notations and definitions

Here, we fix some notations and definitions which are used throughout this paper. For a group H and subgroup $K \subset H$, we let $N_H(K)$ be the normalizer of K in H. For any element $h \in H$, we denote the centralizer of h in H by $C_H(h)$. We define $\underline{H}$ to be the set of conjugacy classes in H. For any $\mathcal{C} \in \underline{H}$, we set $C_H(\mathcal{C})$ to be $C_H(h)$ for any $h \in \mathcal{C}$, which is well-defined up to conjugacy. If H is a topological group, then we set H^0 to be the identity component of H which is a topological subgroup of H.

For a finite dimensional $\mathbb{C}$-algebra A, we denote by $\mathrm{Irr}(A)$ the set of irreducible representations of A over $\mathbb{C}$. If H is a finite group, we write $\mathrm{Irr}(H)$ instead of $\mathrm{Irr}(\mathbb{C}[H])$. Let $Id_H \in \mathrm{Irr}(H)$ be the trivial representation of H. We set $\widehat{H}$ to be the set of all virtual characters of H over $\mathbb{C}$, which is equivalent to the $\mathbb{Z}$-span of $\mathrm{Irr}(H)$. For $h \in H$ and $E \in \widehat{H}$, we denote by $\mathrm{tr}(h, E)$ the character value of E at h. For $\mathcal{C} \in \underline{H}$ define $\mathrm{tr}(\mathcal{C}, E)$ to be $\mathrm{tr}(h, E)$ at any $h \in \mathcal{C}$.

Let $\mathbf{k}$ be an algebraically closed field of characteristic p (which can be zero) and $G = GL_n(\mathbf{k})$. For a variety X over $\mathbf{k}$ and a prime $\ell \neq p$, $\chi(X)$ denotes the (ℓ-adic) Euler characteristic of X defined by the following formula.

$$\chi(X) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\overline{\mathbb{Q}_\ell}} H^i_c(X, \overline{\mathbb{Q}_\ell})$$

(Note that $\sum_{i \in \mathbb{Z}} (-1)^i \dim_{\overline{\mathbb{Q}_\ell}} H^i_c(X, \overline{\mathbb{Q}_\ell}) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\overline{\mathbb{Q}_\ell}} H^i(X, \overline{\mathbb{Q}_\ell})$ by [11].) Also we denote the constant $\overline{\mathbb{Q}_\ell}$-sheaf on X by $\overline{\mathbb{Q}_\ell}_X$.

We fix a standard basis $e_1, \ldots, e_n \in \mathbf{k}^n$ and consider G as the set of invertible $n \times n$ matrices with respect to this fixed basis. Let $B_0 \subset G$ be the subgroup of G consisting of all