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Structure of certain Weyl modules for the Spin groups

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ABSTRACT

Let K be an algebraically closed field of characteristic $p \geqslant 0$ and let W be a finite-dimensional K-vector space of dimension greater than or equal to 5. In this paper, we give the structure of certain Weyl modules for $G = \mathrm{Spin}(W)$ in the case where $p \neq 2$, as well as the dimension of the corresponding irreducible, finite-dimensional, rational KG-modules. In addition, we determine the composition factors of the restriction of certain irreducible, finite-dimensional, rational $K\,\mathrm{SL}(W)$-modules to $\mathrm{SO}(W)$.

1. Introduction

Let K be an algebraically closed field of characteristic $p \geqslant 0$, and let G be a simply connected, simple algebraic group over K. Fixing a Borel subgroup B of G containing a maximal torus T of G, one obtains an associated set of dominant weights for T, denoted by $X^+(T)$. It is well-known that for each $\varpi \in X^+(T)$, there exists a unique (up to isomorphism) finite-dimensional, irreducible, rational KG-module $L_G(\varpi)$ having highest weight ϖ. In other words, the isomorphism classes of finite-dimensional, irreducible, ra-

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tional modules for G are in one-to-one correspondence with the aforementioned dominant weights for T.

In characteristic zero, the dimension of each $L_G(\varpi)$ is known, and is given by the well-known Weyl's degree formula [14, Corollary 24.3]. Also weight multiplicities in $L_G(\varpi)$ can be recursively computed using Freudenthal's formula [11], or one of the many variants developed in the last decades. (We refer the reader to [23], [4], [9], [26], [8], [27], or [6] for a few examples.) Closed formulas can also be used to obtain information on weight multiplicities, or even on the so-called character of a given irreducible module (see [9] or [19], for instance). Observe, however, that those methods are often quite demanding in terms of complexity.

In positive characteristic, not much is known about irreducible KG-modules in general. However, following the construction in [29, Section 2], one obtains a universal highest weight module $V_G(\varpi)$ of highest weight ϖ, for every $\varpi \in X^+(T)$, by finding an appropriate $\mathbb{Z}$-form in a suitable irreducible module for the corresponding complex Lie algebra, and then tensoring it by K. The KG-module $V_G(\varpi)$ is called the *Weyl module* of highest weight ϖ, and has the property that its quotient by its unique maximal submodule $\mathrm{rad}(\varpi)$ is irreducible with highest weight ϖ. In other words, we have

$$L_G(\varpi) \cong V_G(\varpi) \,/\, \mathrm{rad}(\varpi).$$

The formulas introduced above can be used to determine the dimension, the weight multiplicities, and the character of $V_G(\varpi)$. The problem consisting in determining the composition factors of $V_G(\varpi)$, on the other hand, is essentially equivalent to the determination of weight multiplicities in simple modules for G: no closed formula is known to this day, and there seems to be no expectation of finding one in the near future. Although it is possible to proceed in a recursive fashion, by arguing on generating sets for weight spaces, those processes are again quite demanding in terms of complexity, and give no insight on the obtained values. For sufficiently large p and small enough ϖ, other tools are at our disposal, like Kazhdan–Lusztig polynomials [18], or the Jantzen p-sum formula [17, Proposition 8.19]. The former allows one to compute weight multiplicities in a recursive fashion, inspired by the study of Verma modules in characteristic zero [15, Chapter 8]. The latter provides a tool for computing all characters of irreducible modules, but generally only in small rank.

In this paper, we determine the structure of certain Weyl modules for G in the case where $\mathrm{char}\, K \neq 2$ and $G = \mathrm{Spin}(W)$, with W a K-vector space of dimension at least 5. In order to do so, we proceed in two steps: inspired by an idea of McNinch [22], we first determine the composition factors of a well-chosen tilting module for G, in order to reduce the list of possible composition factors for $V_G(\varpi)$, thanks to a generalization of [22, Proposition 4.6.2], namely Proposition 3.2. Finally, a suitable use of a truncated version of the Jantzen p-sum formula (Theorem 3.8) yields the desired result. We then deduce the dimensions of the corresponding irreducible KG-modules, and conclude by proving a result on the composition factors of the restriction to $\mathrm{SO}(W)$ of certain $\mathrm{SL}(W)$-modules.