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Computing maximal subsemigroups of a finite semigroup

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Abstract

A proper subsemigroup of a semigroup is *maximal* if it is not contained in any other proper subsemigroup. A maximal subsemigroup of a finite semigroup has one of a small number of forms, as described in a paper of Graham, Graham, and Rhodes. Determining which of these forms arise in a given finite semigroup is difficult, and no practical mechanism for doing so appears in the literature. We present an algorithm for computing the maximal subsemigroups of a finite semigroup S given knowledge of the Green's structure of S, and the ability to determine maximal subgroups of certain subgroups of S, namely its group $\mathscr{H}$-classes.

In the case of a finite semigroup S represented by a generating set X, in many examples, if it is practical to compute the Green's structure of S from X, then it is also practical to find the maximal subsemigroups of S using the algorithm we present. In such examples, the time taken to determine the Green's structure of S is comparable to that taken to find the maximal subsemigroups. The generating set X for S may consist, for example, of transformations, or partial permutations, of a finite set, or of matrices over a semiring. Algorithms for computing the Green's structure of S from X include the Froidure-Pin Algorithm, and an algorithm of the second author based on the Schreier-Sims algorithm for permutation groups. The worst case complexity of these algorithms is polynomial in $|S|$, which for, say, transformation semigroups is exponential in the number of points on which they act.

Certain aspects of the problem of finding maximal subsemigroups reduce to other well-known computational problems, such as finding all maximal cliques in a graph and computing the maximal subgroups in a group.

The algorithm presented comprises two parts. One part relates to computing the maximal subsemigroups of a special class of semigroups, known as Rees 0-matrix semigroups. The other part involves a careful analysis of certain graphs associated to the semigroup S, which, roughly speaking, capture the essential information about the action of S on its $\mathscr{J}$-classes.

1 Introduction

A *semigroup* S is a set with an associative operation. A *subsemigroup* M of a semigroup S is just a subset that is closed under the operation of S. A *maximal subsemigroup* M of a semigroup S is a proper subsemigroup that is not contained in any other proper subsemigroup. Every proper subsemigroup of a finite semigroup is contained in a maximal subsemigroup, although the same is not true for infinite semigroups. For example, the multiplicative semigroup consisting of the real numbers in the interval $(1, \infty)$ has no maximal subsemigroups, but it does have proper subsemigroups.

There are numerous papers in the literature about finding maximal subsemigroups of particular classes of semigroups; for example [5, 6, 7, 8, 9, 10, 12, 20, 21, 22, 26, 28, 34, 35, 36, 37]. Perhaps the most important paper on this topic for finite semigroups is that by Graham, Graham, and Rhodes [18]. This paper appears to have been overlooked for many years, and indeed, special cases of the results it contains have been repeatedly reproved.

The main purpose of this paper is to give algorithms that can be used to find the maximal subsemigroups of an arbitrary finite semigroup. Our algorithms are based on the paper from 1968 of Graham, Graham, and Rhodes [18]. The algorithms described in this paper are implemented in the GAP [17] package Semigroups [29]. This paper is organised as follows. In the remainder of this section, we introduce the required background material and notation. We state the main results of Graham, Graham, and Rhodes [18] in Propositions 1.4 and 1.5. In Section 2, we describe algorithms for finding the maximal subsemigroups of a finite regular Rees 0-matrix semigroup over a group. In Section 3, we use the procedures from Section 2 to describe an algorithm for finding the maximal subsemigroups of an arbitrary finite semigroup.

Henceforth we only consider finite semigroups, and we let S denote an arbitrary finite semigroup throughout. We denote by S^1 the semigroup obtained from S by adjoining an identity element $1 \notin S$. In other words, $1s = s1 = s$ for all $s \in S^1$. If X is any subset of a semigroup S, then we denote by $\langle X \rangle$ the least subsemigroup of S containing X. If $\langle X \rangle = S$, then we refer to X as a *generating set* for S.

Let $x, y \in S$ be arbitrary. We say that x and y are $\mathscr{L}$-related if the principal left ideals generated by x and y in S are equal; in other words, $S^1x = S^1y$. Clearly $\mathscr{L}$ defines an equivalence relation on S. We write $x\mathscr{L}y$ to denote that x and y