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On the asymptotic linearity of reduction number

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Let R be a standard graded algebra over an infinite field $\mathbb{K}$ and M a finitely generated $\mathbb{Z}$-graded R-module. For any graded ideal $I \subseteq R_+$ of R, we show that the functions $D(I^nM), r(I^nM)$ and $r(M/I^nM)$ are all asymptotically linear. Here $r(\bullet)$ and $D(\bullet)$ stand for the reduction number and the maximal degree of minimal generators of a graded module $\bullet$ respectively.

1. Introduction

Throughout this paper, we always assume that $R = \bigoplus_{n\geq 0} R_n$ is a standard graded Noetherian algebra over an infinite field $\mathbb{K}$. Here "standard graded" means that $R_0 = \mathbb{K}$ and $R = \mathbb{K}[R_1]$. As usual, a nonzero element in R_1 is called a *linear form* of R. Let M be a finitely generated nonzero $\mathbb{Z}$-graded R-module.

Definition 1.1. A graded ideal J of R is called an *M-reduction* if J is generated by linear forms such that $(JM)_n = M_n$ for $n \gg 0$. An M-reduction is called *minimal* if it does

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not contain any other M-reduction. The *reduction number* of M with respect to J is defined to be

$$r_J(M) := \max\{n \in \mathbb{Z}\colon\ (JM)_n \neq M_n\}.$$

The *reduction number* of M is

$$r(M) := \min\{r_J(M)\colon\ J \text{ is a minimal } M\text{-reduction}\}.$$

Let I be a graded ideal of R. In this paper, we are interested in the following natural problem: Is $r(I^nM)$ a linear function of n for all $n \gg 0$? This problem is inspired by the asymptotic behaviour of the Castelnuovo–Mumford regularity $\operatorname{reg}(I^nM)$. It was first shown in [3] and [7] for the case R being a polynomial ring over a field, and then in [10] for the general case (namely, when R is a standard graded algebra over a Noetherian ring with unity) that $\operatorname{reg}(I^nM)$ is a linear function of n for all $n \gg 0$. Since the reduction number $r(I^nM)$ is less than or equal to the Castelnuovo–Mumford regularity $\operatorname{reg}(I^nM)$ [9, Proposition 3.2], it is bounded above by a linear function of n.

A very useful result for investigating the asymptotic property of regularity, which was proved in [10], will play a key role in our research. Let us recall this result. Denote by $S := A[x_1, \dots, x_n, y_1, \dots, y_m]$ the polynomial ring over a commutative Noetherian ring A with unity. We may view S as a bigraded ring with $\deg x_i = (1, 0)$ for $i = 1, \dots, n$ and $\deg y_j = (d_j, 1)$ for $j = 1, \dots, m$. Here $d_1, \dots, d_m$ is a sequence of non-negative integers. Let $\mathcal{U}$ be a finitely generated bigraded module over S. For a fixed number n put

$$\mathcal{U}_n := \bigoplus_{a\in\mathbb{Z}} \mathcal{U}_{(a,n)}.$$

Clearly, $\mathcal{U}_n$ is a finitely generated graded module over the naturally graded ring $A[x_1, \dots, x_n]$. It was proved in [10, Theorem 2.2] that $\operatorname{reg}(\mathcal{U}_n)$ is asymptotically a linear function with slope $\leq \max\{d_1, \dots, d_m\}$. The arguments of our main results are all based on this theorem.

To state our main results, we denote by $D(M)$ and $d(M)$ the largest and least degrees of a minimal system of generators of M respectively. In other words,

$$D(M) := \max\{n \in \mathbb{Z}\colon\ (M/R_+M)_n \neq 0\} \text{ and } d(M) := \min\{n \in \mathbb{Z}\colon\ M_n \neq 0\}.$$

Let I be a graded ideal of R. A graded ideal $J \subseteq I$ is called an *M-reduction of I* if $JI^nM = I^{n+1}M$ for some $n > 0$. Note that we do not require that J is generated by linear forms, hence this concept is very different from the notion of M-reduction as given in Definition 1.1. The integer $\rho_I(M)$ is defined to be

$$\rho_I(M) := \min\{D(J)\colon\ J \text{ is an } M\text{-reduction of } I\}.$$

We first give a positive answer to the above question as follows.