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Base-point-freeness of double-point divisors of smooth birational-divisors on conical rational scrolls

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A B S T R A C T

We work over an algebraically closed field of characteristic zero. The purpose of this paper is to prove that the complete linear system of the double point divisors of smooth birational-divisors on conical rational scrolls are base-point-free. A smooth birational-divisor on a conical rational scroll has a nonbirational inner center, that is a point on it from which the linear projection gives nonbirational map to its image. In the previous paper by the author, it was shown that for a projective variety without nonbirational inner centers, its double point divisor is base-point-free.

0. Introduction

We work over an algebraically closed field $\Bbbk$ of characteristic zero. Let $X \subseteq \mathbb{P}^N$ be a nondegenerate (i.e., not contained in any hyperplane of $\mathbb{P}^N$) smooth projective variety of dimension $n \geq 2$, degree d and codimension $e = N - n$ with the canonical line bundle ω_X. Set $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^N}(1)|_X$. Consider the set $\mathcal{C}(X)$ of points of X from which X is projected nonbirationally onto its image, i.e., $\mathcal{C}(X) = \{u \in X | l(X \cap \langle u, x\rangle) \geq 3$ for general $x \in X\}$. Here $l(Z)$ is the length of a subscheme $Z \subseteq \mathbb{P}^N$ and $\langle\ \rangle$ denotes

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the linear span in $\mathbb{P}^N$. We call such a point a *nonbirational inner center* of X. The purpose of this paper is to show the base-point-freeness of the complete linear system $|\mathcal{O}_X(d-n-e-1)\otimes\omega_X^\vee|$ for some smooth projective variety with nonempty $\mathcal{C}(X)$, which is called a smooth *birational-divisor on a conical rational scroll* $\mathbf{E}_{\mathcal{E}}^P$ defined in (0.2).

Before going to the Main Theorem, we will mention the motivation. For the linear system $|\mathcal{O}_X(d-n-2)\otimes\omega_X^\vee|$, Bayer and Mumford [1] and Bo Ilic [3] studied the base-point-freeness and the separation of distinct two points of X respectively, by looking at divisors of the double-point loci or the ramification loci of projections of X from general $(e-1)$-points of $\mathbb{P}^N \setminus X$ to $\mathbb{P}^{n+1}$. In [5], considering linear projections of X from the linear subspaces spanned by general $(e-1)$-points of X, whose double-point loci are members of $|\mathcal{O}_X(d-n-e-1)\otimes\omega_X^\vee|$, we proved the following theorem.

Theorem 0.1. ([5], Theorems 1 and 3) *The base locus* $\mathrm{Bs}\,|\mathcal{O}_X(d-n-e-1)\otimes\omega_X^\vee|$ *of the line bundle* $\mathcal{O}_X(d-n-e-1)\otimes\omega_X^\vee$ *is contained in* $\mathcal{C}(X)$ *unless* X *is projectively equivalent to a scroll over a curve or the Veronese surface in* $\mathbb{P}^5$.

Here we say that two projective varieties $X \subseteq \mathbb{P}^N$ and $X' \subseteq \mathbb{P}^{N'}$ are *projectively equivalent* if $N = N'$ and if there is an automorphism of $\mathbb{P}^N$ mapping X onto X'.

On the other hand, if $\dim\mathcal{C}(X) \geq 1$, $\mathcal{C}(X)$ contains a line L (see [4], Corollary 6.2) and $\mathcal{O}_X(d-n-2)\otimes\omega_X^\vee|L = \mathcal{O}_L$ (see [3], Proposition 3.8 or [5], Example 6.3(3)), where such a variety X is called a *Roth variety* or its isomorphic image by a linear projection. As a next step, it is natural to ask the line bundle actually has a base-point on $\mathcal{C}(X)$ or not for the case $\dim\mathcal{C}(X) = 0$. One of projective varieties with $\dim\mathcal{C}(X) = 0$ is the following (see §5 for the remaining cases).

Definition 0.2. For integers $\mu \geq 2$, b and $n \geq 2$, a nondegenerate (not necessarily smooth) projective variety $X \subseteq \mathbb{P}^N$ of dimension n and codimension e is said to be a *birational-divisor of type* (μ, b) *on a conical rational scroll* $\mathbf{E}_{\mathcal{E}}^P$ with vertex $P \in \mathbb{P}^N$ for an ample vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ of rank n if X is the birational image $\psi(\tilde{X})$ of an irreducible and reduced divisor $\tilde{X} \in |\mathcal{O}_{\mathbf{E}_{\mathcal{E}}^P}(\mu)\otimes p^*\mathcal{O}_{\mathbb{P}^1}(b)|$ of the projective bundle $\mathbf{E}_{\mathcal{E}}^P := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{E})$ over $\mathbb{P}^1$ such that $P = \psi(\tilde{P})$ for the subbundle $\tilde{P} := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1})(\subseteq \mathbf{E}_{\mathcal{E}}^P)$ by a birational embedding (i.e., birational onto its image) $\psi\colon \mathbf{E}_{\mathcal{E}}^P \to \mathbb{P}^N$ defined by a base-point-free subsystem of $|\mathcal{O}_{\mathbf{E}_{\mathcal{E}}^P}(1)|$. Here $\mathcal{O}_{\mathbf{E}_{\mathcal{E}}^P}(1)$ is the tautological line bundle and $p\colon \mathbf{E}_{\mathcal{E}}^P \to \mathbb{P}^1$ is the projection. Note that $e \leq c_1(\mathcal{E})$ and the equality holds if and only if ψ is defined by the complete linear system $|\mathcal{O}_{\mathbf{E}_{\mathcal{E}}^P}(1)|$. By Grothendieck's theorem, we may assume that $\mathcal{E} \cong \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$ for some positive integers a_i with $a_1 \leq a_2 \leq \cdots \leq a_n$.

The main result here is the following theorem.

Theorem 0.3. *Let* $X \subseteq \mathbb{P}^N$ *be a nondegenerate* n*-dimensional* $(n \geq 2)$ *smooth birational-divisor on a conical rational scroll* $\mathbf{E}_{\mathcal{E}}^P$ *with vertex* $P \in \mathbb{P}^N$ *for an ample vector bundle* $\mathcal{E}$ *on* $\mathbb{P}^1$ *such that* $P \in \mathcal{C}(X)$. *Set* $d := \deg(X)$, $e := \mathrm{codim}(X, \mathbb{P}^N)$ *and* $a := c_1(\mathcal{E})$.