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A new interpretation of the Catalan numbers arising in the theory of crystals

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ABSTRACT

Towards the study of the Kashiwara $B(\infty)$ crystal, sets H^t, $t \in \mathbb{N}$ of functions were introduced given by equivalence classes of unordered partitions satisfying certain boundary conditions [5].

Here it is shown that H^t is a Catalan set of order t, that is to say the cardinality of H^t is the tth Catalan number $\mathcal{C}_t$. This is a new description of a Catalan set and moreover admits some remarkable features.

Thus to H^t there is an associated labelled graph $\mathscr{G}_t$ which is shown to have a *canonical* decomposition into $(t-1)!$ subgraphs each with 2^{t-1} vertices. These subgraphs, called S-graphs, have some tight properties which are needed for the study of $B(\infty)$. They are described as labelled hypercubes in $\mathbb{R}^{t-1}$ whose edges connecting vertices with equal labels are missing.

It is shown that the number of *distinct* hypercubes so obtained is again a Catalan number, namely $\mathcal{C}_{t-1}$. They define functions which depend on a coefficient set of $(t-1)$ non-negative integers. When the latter are non-zero and pairwise distinct, the vertices of the S-graphs describe distinct functions. Moreover this property is retained if certain edges are deleted and certain vertices identified. In particular when these coefficients are all equal and non-zero, it is shown that every hypercube degenerates to a simplex, resulting in exactly t distinct functions, which for example are exactly those needed in the description of $B(\infty)$ in type A.

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Some examples of S-graphs not of the type obtained above are also given.

1. Introduction

Throughout this paper, we denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of natural numbers.

1.1. This paper arose out of the study of the Kashiwara $B(\infty)$ crystal.

For this we introduced [5] certain graphs called S-graphs, a subject examined here. Although this is directed towards the description of $B(\infty)$, we concentrate here on the combinatorics which is of interest in its own right. In particular this involves the Catalan numbers in an entirely new way. The eventual application of S-sets to the study of $B(\infty)$ is outlined in 1.8.

1.2. The Catalan numbers form a sequence of integers; for all $t \in \mathbb{N}$, the tth Catalan number is given by the formula

$$\mathcal{C}_t = \frac{(2t)!}{t!(t+1)!}$$

or, recursively by

$$\mathcal{C}_t = \sum_{j=0}^{t-1} \mathcal{C}_{t-1-j}\mathcal{C}_j,$$

with $\mathcal{C}_0 = 1$.

A set X_t, $t \in \mathbb{N}$ such that $|X_t| = \mathcal{C}_t$ is called a Catalan set (of order t). Of course they nearly always come in families defined for all $t \in \mathbb{N}$. Many examples of Catalan sets are known; the (set of) triangulations of the $(t+2)$-gon, the Dyck paths from $(0, 0)$ to $(0, 2t)$ and the nilpotent ideals in the Borel subalgebra [3] of $\mathfrak{sl}_t$. In [11] one can find 66 examples of Catalan sets and a few more in [12].

1.3. The new example of a Catalan set H^t, which appeared in [5] and we study in this paper, consists of equivalence classes of unordered partitions (Section 2.1) into $t+1$ parts with boundary conditions (Section 2.2). They parametrize certain functions which are involved in an inductive construction of “dual Kashiwara functions”, which in turn are used to describe $B(\infty)$. The observation that the cardinalities of H^t for small t are given by the Catalan numbers is due to S. Zelikson who suggested that this should be true in general. We prove this Zelikson conjecture in Theorem 4.1.