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Simplicities of VOAs associated to Jordan algebras of type B and character formulas for simple quotients

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ABSTRACT

In this paper we study the VOA $V_{\mathcal{J},r}$ constructed by Ashihara and Miyamoto in [2], which satisfies $(V_{\mathcal{J},r})_0 = \mathbb{C}1, (V_{\mathcal{J},r})_1 = \{0\}$, and the Griess algebra $(V_{\mathcal{J},r})_2$ is isomorphic to the type B simple Jordan algebra $\mathcal{J}$. We construct simple quotients $\bar{V}_{\mathcal{J},r}$ for $r \in \mathbb{Z}_{\neq 0}$ using dual-pair type constructions, and we reprove that $V_{\mathcal{J},r}$ is simple if $r \notin \mathbb{Z}$. We also compute the character formula of the simple quotients $\bar{V}_{\mathcal{J},r}$ for $r = -2n, n \in \mathbb{Z}_{>0}$.

1. Introduction to the main results

Let V be a $\mathbb{Z}_{\geq 0}$ graded vertex operator algebra (VOA), with $V_0 = \mathbb{C}1, V_1 = \{0\}$. Then V_2 has a commutative (but not necessarily associative) algebra structure with the operation $a \circ b = a(1)b$. This algebra V_2 is called the Griess algebra of V. In [18] and [19], Lam constructed VOAs whose Griess algebras are simple Jordan algebras of type A, B, and C. For some low rank type D simple Jordan algebras the construction was given by Ashihara in [3]; In [2] Ashihara and Miyamoto constructed a family of VOAs

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$V_{\mathcal{J},r}$ parametrized by $r \in \mathbb{C}$, whose Griess algebras are isomorphic to type B Jordan algebras $\mathcal{J}$. The VOA $V_{\mathcal{J},r}$ was further studied by Niibori and Sagaki in [23].

One of the main results in [23] claims that if $\mathcal{J}$ is not the Jordan algebra of 1×1 matrix, then $V_{\mathcal{J},r}$ is simple if and only if $r \notin \mathbb{Z}$. This suggests that $r \in \mathbb{Z}$ are special and may deserve further study. We show that the simple quotients $\bar{V}_{\mathcal{J},r}, r \in \mathbb{Z}_{\neq 0}$ can be constructed by a dual-pair type construction. We also apply the construction to compute the character $\mathrm{Tr}|_{\bar{V}_{\mathcal{J},r}} q^{L(0)}$ for $r = -2n, n \geq 1$. The Clebsch–Gordan coefficients appear naturally in the character formula.

We give more details about this paper. All vector spaces and Lie groups are assumed to be over $\mathbb{C}$. Let $(\mathfrak{h}, (\cdot,\cdot))$ be a finite dimensional vector space with a non-degenerate symmetric bilinear form $(\cdot,\cdot)$, $\dim(\mathfrak{h}) = d$. Then $\mathfrak{h}\otimes\mathfrak{h}$ has an associative algebra structure:

$$(a \otimes b)(u \otimes v) = (b, u)a \otimes v,$$

which induces a Jordan algebra structure on $\mathfrak{h} \otimes \mathfrak{h}$:

$$x \circ y = \frac{1}{2}(xy + yx),\ \forall x, y \in \mathfrak{h} \otimes \mathfrak{h}.$$

Let $\mathcal{J}$ be the Jordan subalgebra of $\mathfrak{h} \otimes \mathfrak{h}$ consists of symmetric tensors:

$$\mathcal{J} \overset{def.}{=} \mathrm{span}\{L_{a,b} |\, a, b \in \mathfrak{h}\},\ L_{a,b} \overset{def.}{=} a \otimes b + b \otimes a.$$

Then $\mathcal{J}$ is the type B simple Jordan algebra of rank d [11].

In this paper we assume that $d \geq 2$ except when we discuss the character formula in Section 6. Let $V_{\mathcal{J},r}$ be the VOA constructed in [2] and $\bar{V}_{\mathcal{J},r}$ be the corresponding simple quotient. In [23] it is shown that $V_{\mathcal{J},r} = \bar{V}_{\mathcal{J},r}$ if and only if $r \notin \mathbb{Z}$. Our results further show that we can construct $\bar{V}_{\mathcal{J},r}, r \in \mathbb{Z}_{\neq 0}$ explicitly. We divide our constructions into three cases:

Case 1, $r = m, m \geq 1$**:** Let $(V_m, (\cdot,\cdot))$ be a m-dimensional vector space with a non-degenerate symmetric bilinear form. The tensor product space $\mathfrak{h}\otimes V_m$ is a dm-dimensional vector space with the non-degenerate symmetric bilinear form:

$$((a \otimes u), (b \otimes v)) = (a, b)(u, v).$$

Let $\mathcal{H}(\mathfrak{h} \otimes V_m)$ be the Heisenberg VOA associated to the vector space $\mathfrak{h} \otimes V_m$ [8]. The group $O(m)$ acts on the component V_m, therefore it acts as automorphism on $\mathcal{H}(\mathfrak{h}\otimes V_m)$. We construct $\bar{V}_{\mathcal{J},m}$ as:

$$\bar{V}_{\mathcal{J},m} \overset{def.}{=} \mathcal{H}(\mathfrak{h} \otimes V_m)^{O(m)}.$$

Case 2, $r = -2n, n \geq 1$**:** In this case, we need to consider a super vertex operator algebra (SVOA) associated to a symplectic space, which is called the 'symplectic