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Nakayama automorphisms of twisted tensor products

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ABSTRACT

In this paper, we study homological properties of twisted tensor products of connected graded algebras. We focus first on the Ext-algebras of twisted tensor products with a certain form of twisting maps. We show those Ext-algebras are also twisted tensor products, and describe the twisting maps for such Ext-algebras explicitly. With those preparations, we describe Nakayama automorphisms of twisted tensor products of noetherian Artin–Schelter regular algebras.

Introduction

The appearance of twisted tensor product of two algebras provides an impactful solution to the questions about "factorization" and "product" in quantum groups and noncommutative geometry ([3,4,14]). Compared with the common tensor product, the

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twisted tensor product makes up for the limitation of too much commutativity, and is a suitable choice for constructing larger classes of noncommutative algebras. Roughly speaking, the twisted tensor product $A \otimes^{\tau} B$ of algebras A and B is an associative algebra, which is $A \otimes B$ as a vector space with the multiplication defined by a twisting map $\tau : B \otimes A \to A \otimes B$. The notion also has been called R-smash product in several recent papers ([3,11,17]).

For our research interests, A and B are always finitely generated algebras. In order to obtain nice combinatorial and homological properties, it is reasonable to demand that the twisting map τ be determined by the action on the generating set of $B \otimes A$. This requirement can be realized by an algebra homomorphism $\sigma = (\sigma_{ij})$ from A to matrix algebra $\mathbb{M}_m(A)$ and a σ-derivation $\delta = (\delta_1, \cdots, \delta_m)^T$ from A to $A^{\oplus m}$, where m is the number of elements in the minimal generating set of B. In this case, we write $\tau = (\sigma, \delta)$. Significantly, classical Ore extensions and double Ore extensions with zero tails (see [19] for details) are both special cases of such twisted tensor products.

Artin–Schelter regular (AS-regular, for short) algebras play an important role in the study of noncommutative algebras ([1]). Ore extensions and double Ore extensions are commonly used to construct AS-regular algebras. Wang and the first and third named authors used twisted tensor products in the construction of AS-regular algebras in [17]. It was shown that the twisted tensor product of AS-regular algebras A and B preserves AS-regularity in case σ is invertible (see Definition 1.8) and B has a pure resolution, namely, each term of the minimal free resolution of the trivial module concentrates in one degree ([17, Theorem 0.3]). The typical examples of algebras with pure resolutions include (d-)Koszul algebras and piecewise Koszul algebras. The AS-regularity of twisted tensor products inspires us to study more homological properties of this construction.

The Nakayama automorphism is one of important homological invariants for AS-regular algebras, equivalently connected graded skew Calabi–Yau algebras. From the point of view of skew Calabi–Yau categories (see [10]), the Nakayama automorphisms correspond to Serre functors of categories. On the other hand, Nakayama automorphisms control Hopf actions ([5,9]). As special cases, Nakayama automorphisms of Ore extensions and trimmed double Ore extensions of Koszul AS-regular algebras have been studied in [7,20], respectively. Our aim is to give a description of the Nakayama automorphism of the twisted tensor product $A \otimes^{\tau} B$ of noetherian AS-regular algebras A and B, where the graded twisting map $\tau = (\sigma, \delta) : B \otimes A \to A \otimes B$.

To achieve the goal, we start from the study of Ext-algebras of twisted tensor products. The Ext-algebra $E(A) := \underline{\mathrm{Ext}}^*_A(k, k)$ is an incredibly powerful tool used in many areas of mathematics for a connected graded algebra A. We focus on computing the Yoneda product in an Ext-algebra. Let A and B be two connected graded algebras with B having a pure resolution, and let $A \otimes^{\tau} B$ be a twisted tensor product for some graded twisting map $\tau = (\sigma, 0)$. We construct a minimal free resolution of trivial module of $A \otimes^{\tau} B$ (see Proposition 2.10), by the complex obtained in [17, Theorem 0.2]. This free resolution helps us bridge the relationships among the Ext-algebras $E(A)$, $E(B)$ and $E(A \otimes^{\tau} B)$.