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Castelnuovo–Mumford regularity and Ratliff–Rush closure ☆

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ABSTRACT

We establish strong relationships between the Castelnuovo–Mumford regularity and the Ratliff–Rush closure of an ideal. Our results have several interesting consequences on the computation of the Ratliff–Rush closure, the stability of the Ratliff–Rush filtration, the invariance of the reduction number, and the computation of the Castelnuovo–Mumford regularity of the Rees algebra and the fiber ring. In particular, we prove that the Castelnuovo–Mumford regularity of the Rees algebra and of the fiber ring are equal for large classes of monomial ideals in two variables, thereby verifying a conjecture of Eisenbud and Ulrich for these cases.

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Introduction

The Castelnuovo–Mumford regularity is a homological invariant which controls the complexity of graded structures over a polynomial rings [7]. It is well known that the Castelnuovo–Mumford regularity plays an important role in computational Algebraic Geometry and Commutative Algebra [1,36]. It is less known that the Castelnuovo–Mumford regularity can be also defined for graded structures over a commutative ring and that the Castelnuovo–Mumford regularity of the Rees algebra controls the complexity of an ideal; see e.g. [32–34]. The main aim of this paper is to explore this aspect of the Castelnuovo–Mumford regularity by establishing relationships between the Castelnuovo–Mumford regularity of the Rees algebra and the seemingly unrelated Ratliff–Rush closure of an ideal.

The motivation for our work originates from the following conjecture of Eisenbud and Ulrich [8, Conjecture 1.3].

Conjecture. *Let A be a standard graded algebra over a field k. Let I be an $\mathfrak{m}$-primary graded ideal generated by forms of the same degree, where $\mathfrak{m}$ denotes the maximal graded ideal of A. Then* $\operatorname{reg} R(I) = \operatorname{reg} F(I)$*, where $R(I) = \oplus_{n\geq 0} I^n$ is the Rees algebra and $F(I) = \oplus_{n\geq 0} I^n/\mathfrak{m}I^n$ is the fiber ring of I.*

For an arbitrary graded ideal I, it is known that $\operatorname{reg} I^n$ is asymptotically a linear function [4,19,35]. However, very little is known on the stability index of $\operatorname{reg} I^n$, i.e. the least number n where $\operatorname{reg} I^n$ becomes a linear function afterward. Eisenbud and Ulrich [8] showed that under the above assumption, $\operatorname{reg} I^n$ is related to the presentation of $R(I)$ as a direct sum of modules over $F(I)$, which has led them to raise the above conjecture.

The conjecture of Eisenbud and Ulrich is not true if one does not put further assumption on the base ring A. We shall see that if the conjecture were true, then A must be a Buchsbaum ring. In fact, Ulrich communicated to the authors that the conjecture should be formulated for polynomial rings. To give an answer to this modified conjecture seems to be difficult because there were no tools, which allow us to compare $\operatorname{reg} R(I)$ and $\operatorname{reg} F(I)$. Note that $R(I)$ also has an $\mathbb{N}$-graded structure over k, which has a different Castelnuovo–Mumford regularity. This Castelnuovo–Mumford regularity was studied by Herzog, Popescu and Trung in [14].

We shall see in this paper that superficial sequences of I can be used as a common tool to characterize $\operatorname{reg} R(I)$ and $\operatorname{reg} F(I)$. As a first consequence, we show that $\operatorname{reg} R(I) = \operatorname{reg} F(I)$ provided $\operatorname{depth} G(I) \geq \dim A - 1$, where $G(I) := \oplus_{n\geq 0} I^n/I^{n+1}$ is the associated graded ring of I. Our main results are however the findings that $\operatorname{reg} R(I)$ and $\operatorname{reg} F(I)$ control the behavior of the Ratliff–Rush closure and the Ratliff–Rush filtration of I.