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ON MINIMAL FREE RESOLUTIONS OF SUB-PERMANENTS AND OTHER IDEALS ARISING IN COMPLEXITY THEORY

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Abstract. We compute the linear strand of the minimal free resolution of the ideal generated by $k \times k$ sub-permanents of an $n \times n$ generic matrix and of the ideal generated by square-free monomials of degree k. The latter calculation gives the full minimal free resolution by [1]. Our motivation is to lay groundwork for the use of commutative algebra in algebraic complexity theory. We also compute several Hilbert functions relevant for complexity theory.

1. Introduction

We study homological properties of two families of ideals over polynomial rings: the ideals $\mathcal{I}^{sqf;n,k} \subset \mathbb{C}[x_1, \dots, x_n]$ generated by square-free monomials of degree k in n variables and the ideals $\mathcal{I}^{perm_n,k} \subset \mathbb{C}[x_{i,j}]_{1 \le i,j \le n}$ generated by $k \times k$ sub-permanents of an $n \times n$ generic matrix. Recall that the permanent of an $m \times m$ matrix $Y = (y_{i,j})$ is the polynomial

$$perm_m(Y) = \sum_{\sigma \in \mathfrak{S}_m} y_{1,\sigma(1)} y_{2,\sigma(2)} \cdots y_{m,\sigma(m)},$$

where $\mathfrak{S}_m$ denotes the symmetric group on m elements.

We obtain our results via larger ideals $I_{1\times k}(1,n)$ (resp. $I_{1\times k}(n,n)$). The ideal $I_{1\times k}(1,n)$ is generated by all monomials of degree k in n variables. The ideal $I_{1\times k}(n,n)$ is generated by permanents of $k \times k$ matrices produced from X where repetition of rows and columns is allowed. Invariantly, $I_{1\times k}(1,n) = \oplus_{j \ge k} S^j \mathbb{C}^n$ is the ideal generated by $S^k\mathbb{C}^n$ and $I_{1\times k}(n,n) \subset Sym(\mathbb{C}^{n^2})$ is the ideal generated $S^k\mathbb{C}^n \otimes S^k\mathbb{C}^n \subset S^k(\mathbb{C}^n \otimes \mathbb{C}^n)$. The main result in each case says that the linear strand of resolution of $\mathcal{I}^{sqf;n,k}$ (resp. $\mathcal{I}^{perm_n,k}$) is the subcomplex of the linear strand of the resolution of $I_{1\times k}(1,n)$ (resp. of $I_{1\times k}(n,n)$) consisting of elements of *regular weights* (cf. §2).

Our motivation comes from complexity theory. We seek to find differences between the homological behavior of ideals generated by $k \times k$ minors (i.e., subdeterminants) of the generic matrix and the ideals generated by $k \times k$ subpermanents. The ideal generated by square-free monomials arises as the $(n-k)$-th Jacobian ideal of the monomial $x_1 x_2 \dots x_n$.

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