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Lattice extensions of Hecke algebras

Ivan Marin

LAMFA, UMR CNRS 7352, Université de Picardie-Jules Verne, Amiens, France

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We investigate the extensions of the Hecke algebras of finite (complex) reflection groups by lattices of reflection subgroups that we introduced, for some of them, in our previous work on the Yokonuma–Hecke algebras and their connections with Artin groups. When the Hecke algebra is attached to the symmetric group, and the lattice contains all reflection subgroups, then these algebras are the diagram algebras of braids and ties of Aicardi and Juyumaya. We prove a structure theorem for these algebras, generalizing a result of Espinoza and Ryom-Hansen from the case of the symmetric group to the general case. We prove that these algebras are symmetric algebras at least when W is a Coxeter group, and in general under the trace conjecture of Broué, Malle and Michel.

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E-mail address: ivan.marin@u-picardie.fr.

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1. Introduction

Let W be finite complex reflection group, for instance a finite Coxeter group. Let B denote the braid group associated to W in the sense of Broué–Malle–Rouquier (see [8]), which in the case of a finite Coxeter group coincides with the Artin group attached to it. We denote $\pi : B \to W$ the natural projection.

The object of this paper is to introduce and analyze a family of algebras denoted $\mathcal{C}(W, \mathcal{L})$, where $\mathcal{L}$ is a finite join semi-lattice which lies inside the poset made of the full reflection subgroups of W, ordered by inclusion. Here a reflection subgroup of W is called *full* if, for any reflection in this subgroup, all the (pseudo-)reflections with the same reflecting hyperplane belong to it. The semi-lattice $\mathcal{L}$ is additionally supposed to be stable under the natural action of W on the lattice of reflection subgroups, and to contain all the cyclic (full) reflection subgroups, and the trivial subgroup as well. Such a semi-lattice will be called an *admissible* semi-lattice.

Let $\mathcal{A}$ denote the hyperplane arrangement attached to W, namely the collection of its reflecting hyperplanes. Let $\mathbf{k}$ be a commutative ring with 1, containing elements $a_{H,i}$ where $H \in \mathcal{A}$, $0 \leq i < m_H$ where m_H is the order of the cyclic subgroup of W fixing H, with the convention that $a_{H,i} = a_{w(H),i}$ for every $H \in \mathcal{A}, w \in W$ and $a_{H,0}$ is invertible inside $\mathbf{k}$. Let R denote the generic ring of Laurent polynomials with integer coefficients $\mathbf{Z}[a_{H,i}, a_{H,0}^{\pm 1}]$, with the same conventions. Our conditions on $\mathbf{k}$ mean that it is a R-algebra. We now define $\mathbf{k}$-algebras $\mathcal{C}_{\mathbf{k}}(W, \mathcal{L})$, with the convention that $\mathcal{C}(W, \mathcal{L}) = \mathcal{C}_R(W, \mathcal{L})$.

These algebras are defined as follows. First consider the algebra $\mathbf{k}\mathcal{L}$ defined as the free $\mathbf{k}$-module with basis elements $e_\lambda, \lambda \in \mathcal{L}$, and where the multiplication is defined by $e_\lambda e_\mu = e_{\lambda \vee \mu}$. This is sometimes called the Möbius algebra of $\mathcal{L}$. Elements of $\mathcal{L}$ can be identified with the collection of reflecting hyperplanes attached to them, and we let $e_H = e_{\{H\}}$ denote the idempotent attached to the subgroup fixing $H \in \mathcal{A}$. We shall use this identification whenever it is convenient to us.

By definition W acts by automorphisms on $\mathbf{k}\mathcal{L}$, hence so does B, and one can form the semidirect product $\mathbf{k}B \ltimes \mathbf{k}\mathcal{L}$. The algebras $\mathcal{C}_{\mathbf{k}}(W, \mathcal{L})$ are defined as the quotient of $\mathbf{k}B \ltimes \mathbf{k}\mathcal{L}$ by the two-sided ideal $\mathfrak{J}$ generated by the elements $\sigma^{m_H} - 1 - Q_s(\sigma) e_H$ where σ runs among the braided reflections of B, $s = \pi(\sigma)$ is the corresponding pseudo-reflection, $H = \mathrm{Ker}\,(s-1)$, and $Q_s(X) = \sum_{k=0}^{m_H - 1} a_{H,k} X^k - 1 \in \mathbf{k}[X]$ (see section 2.3.2 for more details).