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Canonical bases of invariant polynomials for the irreducible reflection groups of types E_6, E_7, and E_8

Vittorino Talamini [a,b,*]

[a] DMIF, Università di Udine, via delle Scienze 206, 33100 Udine, Italy
[b] INFN, Sezione di Trieste, via Valerio 2, 34127 Trieste, Italy

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A B S T R A C T

Given a rank n irreducible finite reflection group W, the W-invariant polynomial functions defined in $\mathbb{R}^n$ can be written as polynomials of n algebraically independent homogeneous polynomial functions, $p_1(x), \ldots, p_n(x)$, called basic invariant polynomials. Their degrees are well known and typical of the given group W. The polynomial $p_1(x)$ has the lowest degree, equal to 2. It has been proved that it is possible to choose all the other $n-1$ basic invariant polynomials in such a way that they satisfy a certain system of differential equations, including the Laplace equations $\triangle p_a(x) = 0$, $a = 2, \ldots, n$, and so are harmonic functions. Bases of this kind are called canonical. Explicit formulas for canonical bases of invariant polynomials have been found for all irreducible finite reflection groups, except for those of types E_6, E_7 and E_8. Those for the groups of types E_6, E_7 and E_8 are determined in this article.

* Correspondence to: DMIF, Università di Udine, via delle Scienze 206, 33100 Udine, Italy.
E-mail address: vittorino.talamini@uniud.it.

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1. Introduction

When a rank n irreducible finite reflection group W acts in $\mathbb{R}^n$, there are no fixed points besides the origin of $\mathbb{R}^n$, and, with no loss of generality, one can assume that $W \subset O(n)$. By saying that $f(x)$, $x \in \mathbb{R}^n$ is a *W-invariant function*, one means that $f(gx) = f(x)$, for all $g \in W$, $x \in \mathbb{R}^n$. It is well known from Ref. [1] that there exist a basis of n algebraically independent W-invariant real homogeneous polynomial functions $p_1(x), \ldots, p_n(x)$, called *basic invariant polynomials*, such that all W-invariant polynomial functions can be univocally written as polynomial functions of the basic invariant polynomials: $f(x) = \widehat{f}(p_1(x), \ldots, p_n(x))$, with $\widehat{f}(p_1, \ldots, p_n) = \widehat{f}(p)$ a polynomial function of $p \in \mathbb{R}^n$. We recall that, because of the algebraic independence of $p_1(x), \ldots, p_n(x)$, there is no polynomial $\widehat{f}(p)$, not equal to the null polynomial 0 (that one with all vanishing coefficients), for which $\widehat{f}(p_1(x), \ldots, p_n(x)) = 0$, for all $x \in \mathbb{R}^n$.

Given the irreducible finite reflection group W, there are infinitely many possible choices of a set of n basic invariant polynomials, but their degrees $d_a = \deg(p_a(x))$, $a = 1, \ldots, n$, are well known and typical of the given group W. They were determined by Coxeter in Ref. [2], and they are all different, except in the case of the groups of type D_n, with even n, in which case there are two basic invariant polynomials of degree n. Usually, as we do, the basic invariant polynomials are ordered according to their degrees, by requiring

$$d_1 \leq d_2 \leq \ldots \leq d_n \,.$$

The irreducibility of W implies $d_1 = 2$, and the orthogonality of W allows one to take

$$p_1(x) = \sum_{i=1}^{n} {x_i}^2 \,. \tag{1}$$

The other $n-1$ basic invariant polynomials have expressions that depend on the specific group, but are not univocally determined. To be more precise, a rank n finite orthogonal reflection group W is generated by the n orthogonal reflections with respect to the hyperplanes orthogonal to the n simple roots. A rotation of the n simple roots with respect to the system of coordinates used in $\mathbb{R}^n$, imply different matrix expressions for the group elements and these imply different explicit expressions of the basic invariant polynomials of degree greater than 2. Moreover, even if the matrix representation of the group W is given, the choice of the basic invariant polynomials is not unique, because it is possible to make algebraic combinations of a given set of basic polynomials to obtain a new set of basic invariant polynomials equally good for that matrix representation (such a change of basis is described in detail in step 2 in Section 2).

Several authors determined explicit bases of invariant polynomials for (some of) the irreducible finite reflection groups, for example Coxeter in Ref. [2], Ignatenko in Ref. [6], Mehta in Ref. [9], and many others that are not cited here. A few authors tried to select, among the infinitely many possibilities, some distinguished bases of invariant

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