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POOR MODULES WITH NO PROPER POOR DIRECT SUMMANDS

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Abstract. As a mean to provide intrinsic characterizations of poor modules, the notion of a pauper module is introduced. A module is a pauper if it is poor and has no proper poor direct summand. We show that not all rings have pauper modules and explore conditions for their existence. In addition, we ponder the role of paupers in the characterization of poor modules over those rings that do have them by considering two possible types of ubiquity: one according to which every poor module contains a pauper direct summand and a second one according to which every poor module contains a pauper as a pure submodule. The second condition holds for the ring of integers and is just as significant as the first one for Noetherian rings since, in that context, modules having poor pure submodules must themselves be poor.

It is shown that the existence of paupers is equivalent to the Noetherian condition for rings with no middle class. As indecomposable poor modules are pauper, we study rings with no indecomposable right middle class (i.e. the ring whose indecomposable right modules are pauper or injective.) We show that semiartinian V-rings satisfy this property and also that a commutative Noetherian ring R has no indecomposable middle class if and only if R is the direct product of finitely many fields and at most one ring of composition length 2. Structure theorems are also provided for rings without indecomposable middle class when the rings are Artinian serial or right Artinian.

Rings for which not having an indecomposable middle class suffices not to have a middle class include commutative Noetherian and Artinian serial rings. The structure of poor modules is completely determined over commutative hereditary Noetherian rings. Pauper Abelian groups with torsion-free rank one are fully characterized.

1. Introduction

While poor modules have been studied from the perspective of the injective profile of the ring, an intrinsic characterization of poor modules is also desirable. An immediate obstacle is that, since the injectivity domain of the direct sum of two modules is the intersection of the domains of injectivity of the summands, any module having a poor module as a direct summand will itself be poor. So, there might not be, in principle, anything interesting about the structure of certain summands and one must focus on those summands that are inherently poor. That leads us to the notion of a *pauper*. A module is a pauper (or a pauper module) if it is poor and no proper direct summand of it is poor.

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