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Morita equivalence and quotient rings

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Let A and B be rings and let M be an A–B-bimodule that is finitely generated and projective in A-mod and in mod-B. Also let I be an ideal of A and let J be an ideal of B such that $IM = MJ$. Our main result is a partial converse of a known result:

Proposition. *Suppose that $I \leq J(A)$, $J \leq J(B)$ so that $M/(IM)$ is an $\overline{A} = A/I$–$\overline{B} = B/J$-bimodule that is finitely generated and projective in $\overline{A}$-mod and in mod-$\overline{B}$ and that induces a Morita Equivalence between $\overline{A}$-mod and $\overline{B}$-mod. Then M induces a Morita Equivalence between A-mod and B-mod.*

This result should be particularly useful in the context that A and B are $\mathcal{O}$-algebras where $\mathcal{O}$ is a commutative local ring, $I = J(\mathcal{O})A$ and $J = I(\mathcal{O})B$. In which case, $\overline{A}$ and $\overline{B}$ are finite dimensional algebras over the field $k = \mathcal{O}/J(\mathcal{O})$.

1. Introduction and main results

Our notation and terminology are standard and tend to follow [1]. All rings have identities and all modules over a ring are unitary.

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Let R be a ring. Then R-mod (resp. mod-R) will denote the category of left (resp. right) R-modules.

In this section, we state and prove our main result (Proposition 1.1). In Section 2, we prove several results that are required in our proof of Proposition 1.1.

Let A and B be rings and let M be an A–B-bimodule such that M is finitely generated and projective in A-mod and in mod-B. Also let I be an ideal of A and let J be an ideal of B such that $IM = MJ$. Then, it is well-known that, if M induces a Morita equivalence between A-mod and B-mod, then $M/(IM)$ induces Morita equivalence between the rings A/I and B/J (cf. [1, Proposition 21.11]).

Our main result is a partial converse to this result:

Proposition 1.1. *Assume the hypotheses above and that $I \leq J(A)$ and $J \leq I(B)$ and suppose also that $M/(IM)$ induces a Morita equivalence between A/I and B/J. Then M induces a Morita equivalence between A and B.*

Remark 1.2. Suppose that $\mathcal{O}$ is a commutative ring, that A and B are finitely $\mathcal{O}$-algebras, and that M is an A–B-bimodule such that $\alpha m = m\alpha$ for all $m \in M$, and $\alpha \in \mathcal{O}$. Let I be an ideal of $\mathcal{O}$ contained in $J(\mathcal{O})$ so that $IM = MI$, $IA = AI$ is an ideal in A and $IB = BI$ is an ideal in B. Assume also that M is finitely generated over $\mathcal{O}$ and hence ${}_AM$ is finitely generated in A-mod and M_B is finitely generated in mod-B. Assume also that M is a projective in A-mod and in mod-B. Then, of course, Proposition 1.1 applies.

We proved Proposition 1.1 in this context in a previous version. A very astute referee suggested that similar arguments might avail to prove a generalization (Proposition 1.1) of our $\mathcal{O}$-algebra investigations.

Remark 1.3. In the case that A and B are $\mathcal{O}$-algebras with $\mathcal{O}$ a commutative local ring, then $k = \mathcal{O}/J(\mathcal{O})$ is a field. Set $I = J(\mathcal{O})$. Then Proposition 1.1 reduces, under the hypotheses of Proposition 1.1, a proof of the Morita Equivalence of A and B to the "$\overline{A}$ and $\overline{B}$ are finite dimensional algebras over the field k" case. This particular issue arose in [2, Proposition 4.14.5].

A proof of proposition. Assume the hypotheses of Proposition 1.1. Here $M/(IM) = M/(MJ)$ is a projective generator in A/I-mod and in mod-B/J (cf. [1, Theorem 22.1]).

Thus Lemma 2.1 implies that M is a finitely generated projective generator in mod-B. Thus $\mathrm{End}_A(M) = \mathcal{H}om_A(M, M)$ is a finitely generated module in mod-B by Lemma 2.2.

Let $\pi_B : B \to B/J$ denote the canonic ring epimorphism and let $\Gamma : \mathcal{H}om_A(M, M) = \mathrm{End}_A(M) \to \mathcal{H}om_A(M/(IM), M(IM))$ denote the ring homomorphism such that $\Gamma(f)(m + IM) = f(m) + IM$ for all $m \in M$ and all $f \in \mathcal{H}om_A(M, M)$. Let $\rho_A : B^{op} \to \mathrm{End}_A(M)$ denote the ring homomorphism such that $\rho_B(b)(m) = mb$ for all $m \in M$ and $b \in B$ and similarly define $\rho_{B/J} : (B/I)^{op} \to \mathrm{End}_{A/I}(M/(IM))$. By [1, Theorem 17.8(i)] both ρ_B and $\rho_{B/I}$ are injective.