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Products of ideals and jet schemes

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ABSTRACT

In the present paper, we give a full description of the jet schemes of the polynomial ideal $(x_1 \cdots x_n) \in k[x_1, \ldots, x_n]$ over a field of zero characteristic. We use this description to answer questions about products and intersections of ideals emerged recently in algorithmic studies of algebraic differential equations.

1. Introduction

Properties of ideals in rings of differential polynomials are essential for the algorithmic study of algebraic differential equations [3,15]. The following fact is an important part of the recent first bound for the effective differential elimination ([15]). If $h_1, \ldots, h_n$ are natural numbers, then there exists d such that for every ideals $I_1, \ldots, I_n$ in a commutative associative differential algebra over a field of zero characteristic

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$$\left(I_1^{(h_1)} \cdots I_n^{(h_n)}\right)^d \subset (I_1 \cdots I_n)^{(h_1+\ldots+h_n)}, \tag{1.1}$$

where, for an ideal I, we denote the ideal generated by derivatives of elements of I of order at most h by $I^{(h)}$. Such an inclusion allows us to reduce a problem about an arbitrary ideal to the problem about an ideal with additional useful properties (for example, to prime ideals in [15]). This kind of reduction is expected to have many potential applications to the algorithmic problems about algebraic differential equations. The value of such d was not needed for obtaining the bound in [15], but we expect that it can be used for refining this and related bounds similarly to the way the Noether exponent was used in [3, Lemma 3.1] and [15, Lemma 5.5]. The problem of determining the number d in (1.1) as well as many other questions about products of ideals in differential algebras can be reduced to questions about the jet ideal of the polynomial ideal generated by $x_1 \cdots x_n \in k[x_1, \ldots, x_n]$ (see [15, Lemma 6.2]).

Recently, jet schemes were successfully applied to study singularities of algebraic varieties (for example, [14]). In [1,16], jet ideals of the ideal generated by $x_1 \cdots x_n$ were studied, minimal primes and their multiplicities were found. In the context of differential algebra, jet schemes and related objects proved to be one of main tools in differential algebraic geometry (see [12,8]).

In the present paper, we generalize results of [1,16] giving a full description of jet schemes of the ideal generated by $x_1 \cdots x_n$ (Section 4), where an unexpected connection to Shubert calculus occurs (see Proposition 3.10). Then we use the obtained information in order to solve the original problem about differential algebraic equations, namely find the minimal possible number d in (1.1) (see Corollary 5.2 and Corollary 5.4).

We also note that our problem is connected to a classical membership problem for the differential ideal generated by $x_1 \ldots x_n$ (see [9,5,6]). In particular, if a differential polynomial belongs to any of the jet ideals of the polynomial ideal generated by $x_1 \cdots x_n$, then it belongs to the corresponding differential ideal, and if a differential polynomial does not belong to any of these jet ideals, it does not belong to the differential ideal. Our structural results give an effective way for deciding if any of these situations takes place (see Remark 5.5).

2. Preliminaries

2.1. Differential algebra

Throughout the paper, all fields are assumed to be of characteristic zero. Unless otherwise stated, all algebras are commutative, associative, and with unity.

Let R be a ring. A map $D\colon R \to R$ satisfying $D(a + b) = D(a) + D(b)$ and $D(ab) = aD(b) + D(a)b$ for all $a, b \in R$ is called a *derivation*. A *differential ring* R is a ring with a specified derivation. In this case, we will denote $D(x)$ by x' and $D^n(x)$ by $x^{(n)}$. A differential ring that is a field will be called a *differential field*.