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# Relative cohomology of complexes ☆

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Let $R$ be an arbitrary ring and $C$ a complex with finite Gorenstein projective dimension (that is, the supremum of Gorenstein projective dimension of all $R$-modules in $C$ is finite). For each complex $D$, we define the $n$th relative cohomology group $\mathrm{Ext}^n_{\mathcal{GP}}(C, D)$ by the equality $\mathrm{Ext}^n_{\mathcal{GP}}(C, D) = \mathrm{H}^n\mathcal{H}\mathrm{om}(G, D)$, where $G \longrightarrow C$ is a strict Gorenstein projective precover of $C$. If $D$ is a complex with finite Gorenstein injective dimension (that is, the supremum of Gorenstein injective dimension of all $R$-modules in $D$ is finite), then one can use a dual argument to define a notion of relative cohomology group $\mathrm{Ext}^n_{\mathcal{GI}}(C, D)$. We show that if $C$ is a complex with finite Gorenstein projective dimension and $D$ a complex with finite Gorenstein injective dimension, then for each $n \in \mathbb{Z}$ there exists an isomorphism $\mathrm{Ext}^n_{\mathcal{GP}}(C, D) \cong \mathrm{Ext}^n_{\mathcal{GI}}(C, D)$. This shows that the relative cohomology functor of complexes is balanced. Some induced exact sequences concerning relative cohomology groups are considered.



## 1. Introduction

The relative cohomology theory was introduced by Eilenberg and Moore in their 1965 AMS Memoir [7] and further studied by MacLane in [14]. When $R$ is a two-sided

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Noetherian ring and a left $R$-module $M$ admits a proper resolution $G \longrightarrow M$ by finite modules of Gorenstein dimension 0, Avramov and Martsinkovsky [4] associate with $M$, for each $n \in \mathbb{Z}$ and each $R$-module $N$, a relative cohomology group $\mathrm{Ext}^n_{\mathcal{G}}(M, N)$ by setting $\mathrm{Ext}^n_{\mathcal{G}}(M, N) = \mathrm{H}^n\mathrm{Hom}_R(G, N)$. For an arbitrary associative ring $R$, and a left $R$-module $M$ that has a proper left Gorenstein projective resolution $\cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$, Holm in [10] defined a relative cohomology group $\mathrm{Ext}^n_{\mathcal{GP}}(M, N)$ by setting $\mathrm{Ext}^n_{\mathcal{GP}}(M, N) = \mathrm{H}^n\mathrm{Hom}_R(G, N)$ for each $n \in \mathbb{Z}$ and each $R$-module $N$. Holm also considered a relative cohomology group $\mathrm{Ext}^n_{\mathcal{GI}}(M, N)$ by a similar method and dealt with the balance of these functors. The relative cohomology groups based on Gorenstein injective modules and the question of Gorenstein balance were already considered by Asadollahi and Salarian in [1]. Sather-Wagstaff, Sharif and White considered relative cohomology theories in [15] and [16] on subcategories of abelian categories and with respect to a semidualizing module, respectively. Veliche [17, Sec. 6] generalized the most results of [4] to the more general set-up of [10].

It is natural to ask an extended version of relative cohomology theory to the setting of complexes. Veliche [17, Remark 6.7] noted that although it is tempting to define the relative cohomology modules using $\mathrm{H}^n\mathcal{H}\mathrm{om}(G, D)$ (where $G$ is a special Gorenstein projective resolution of $C$ in the sense of [17]), this is not suitable because we do not know whether this construction has the necessary uniqueness and functoriality. When $C$ is a complex of finite Gorenstein projective dimension in the sense of Veliche [17] and $D$ is an $R$-module, Asadollahi and Salarian in [2] used cohomology groups of the total complex $\mathcal{H}\mathrm{om}(P_C, P_D)_{ba}$, to define a notion of relative Ext, denoted as $\mathrm{Ext}^n_{\mathcal{GP}}(C, D)$, where $P_C$ (resp. $P_D$) is a DG-projective resolution of $C$ (resp. $D$) and the subscript $ba$ applied to the Hom functor $\mathcal{H}\mathrm{om}(P_C, P_D)$ serves for bounded above morphisms. Iacob [11] used the fact that over a Gorenstein ring every complex has a special Gorenstein projective precover (by [8]) to define a notion of relative cohomology: if $G \longrightarrow C$ is a special Gorenstein projective precover of $C$ then for a complex $D$ the $n$th Gorenstein cohomology group is defined by $\mathrm{Ext}^n_{\mathcal{G}}(C, D) = \mathrm{H}^n\mathcal{H}\mathrm{om}(G, D)$. In this paper we will consider relative cohomology theory of complexes over arbitrary associative rings. Let $R$ be an arbitrary ring and $C$ a complex with finite Gorenstein projective dimension in the sense of [13] (that is, the supremum of Gorenstein projective dimension of all $R$-modules in $C$ is finite, which implies that $C$ has a strict Gorenstein projective precover). For each complex $D$, we define the $n$th relative cohomology group $\mathrm{Ext}^n_{\mathcal{GP}}(C, D)$ by the equality

$$\mathrm{Ext}^n_{\mathcal{GP}}(C, D) = \mathrm{H}^n\mathcal{H}\mathrm{om}(G, D),$$

where $G \longrightarrow C$ is a strict Gorenstein projective precover of $C$.

If $D$ is a complex with finite Gorenstein injective dimension in the sense of [12], then $D$ has a strict Gorenstein injective preenvelope. So one can use a dual argument to define a notion of relative cohomology: if $D \longrightarrow H$ is a strict Gorenstein injective preenvelope then for each complex $C$, one can define the $n$th relative cohomology group $\mathrm{Ext}^n_{\mathcal{GI}}(C, D)$ by the equality $\mathrm{Ext}^n_{\mathcal{GI}}(C, D) = \mathrm{H}^n\mathcal{H}\mathrm{om}(C, H)$. In the case of modules Holm