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Generalized μ-ordinary Hasse invariants

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ABSTRACT

We give a short proof for the existence of μ-ordinary Hasse invariants for the good reduction special fiber of Shimura varieties of Hodge-type using the stack of G-zips introduced by Moonen–Wedhorn and Pink–Wedhorn–Ziegler. We give an explicit formula for the power of the Hodge bundle that admits a Hasse invariant. When G is a Weil restriction of G_1, we relate the Ekedahl–Oort strata of G and those of G_1.

Introduction

This article predates the recent preprints [6] and [7] of the first author joint with W. Goldring on the existence of Hasse invariants. See below for a comparative description of these papers.

Shimura varieties and G-zips

Let $(\mathbf{G}, X)$ be a Shimura datum of Hodge-type and let $\mathcal{S}_K$ be the Kisin–Vasiu integral model of the associated Shimura variety $Sh_K(\mathbf{G}, X)$ at a level K, hyperspecial at p. Denote by S_K the special fiber of $\mathcal{S}_K$ and write G for the special fiber of a reductive $\mathbb{Z}_p$-model of $\mathbf{G}_{\mathbb{Q}_p}$.

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Recall that Zhang [22] gives a smooth morphism $\zeta : S_K \to G\text{-}\mathtt{Zip}^\mu$, where $G\text{-}\mathtt{Zip}^\mu$ is the stack of G-zips, defined by Pink–Wedhorn–Ziegler in [18] (see also the precursor paper [16]) and μ pertains to the cocharacter attached to the Shimura datum. The fibers of ζ are termed Ekedahl–Oort strata of S_K. In this paper we study the open zip stratum $U_\mu \subset G\text{-}\mathtt{Zip}^\mu$ and its corresponding generic Ekedahl–Oort stratum $S_{K,\mu} \subset S_K$. It coincides with the μ-ordinary Newton stratum [21].

Attached to the pair (G, μ), there is a zip datum $\mathcal{Z} := (G, P, L, Q, M, \varphi)$ (§1.4), where L is the centralizer of μ in G and P corresponds to the stabilizer of the Hodge filtration. One attaches to each $\lambda \in X^*(L)$ a line bundle $\mathscr{V}(\lambda)$ on the stack $G\text{-}\mathtt{Zip}^\mu$. Its pull-back $\zeta^*(\mathscr{V}(\lambda))$ coincides with the automorphic line bundle $\mathscr{V}_K(\lambda)$ naturally attached to λ. For example, there exists $\lambda_\omega \in X^*(L)$ such that $\mathscr{V}_K(\lambda_\omega)$ is the Hodge line bundle ω on S_K.

Hasse invariants

In this paper, we say that a section $h \in H^0(G\text{-}\mathtt{Zip}^\mu, \mathscr{V}(\lambda))$ is a Hasse invariant if its non-vanishing locus is exactly the μ-ordinary stratum U_μ. There is an explicit integer N_μ (Definition 3.2.4) satisfying the following:

Theorem 1 *(Theorem 5.1.4). If $\lambda \in X^*(L)$ is $\mathcal{Z}$-ample, there exists a Hasse invariant $h \in H^0(G\text{-}\mathtt{Zip}^\mu, \mathscr{V}(N_\mu \lambda))$.*

For the definition of $\mathcal{Z}$-ample, see Definition 5.1.1. The character λ_ω defining the Hodge line bundle is $\mathcal{Z}$-ample. In [6, Th. 3.2.3], a similar result for all strata is proved using a group-theoretical counterpart of a flag space of Ekedahl–Van der Geer. The methods used here to prove Theorem 1 differ in many aspects from [6]; they are based on the study of Ekedahl–Oort strata in the case of a Weil restriction, which we explain below.

Furthermore, we want to point out that we do not assume in Theorem 1 that λ satisfies the condition “orbitally p-close” of [6, Th. 3.2.3]. Hence Theorem 1 gives a stronger result than [6, Th. 3.2.3] for the open zip stratum. Another improvement is the fact that we determine explicitly the integer N_μ in Theorem 1, whereas [6, Th. 3.2.3] gives an undetermined integer. This is the smallest integer satisfying the existence of a Hasse invariant.

In particular, we obtain the following corollary (Corollary 5.6.1):

Corollary 1. *There exists a section $h_K \in H^0(S_K, \omega^{N_\mu})$ whose non-vanishing locus is the μ-ordinary locus of S_K.*

Prior to the present article, Hasse invariants were constructed by a number of authors: E. Goren established the existence of partial Hasse invariants for the case of Hilbert–Blumenthal Shimura varieties [9]. In the split unitary case of signature $(n-1, 1)$, Ito constructed Hasse invariants for all Ekedahl–Oort strata [11]. Finally, Goldring and Nicole constructed a μ-ordinary Hasse invariant for unitary Shimura varieties [8].