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Triangulated quotient categories revisited ☆

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ABSTRACT

Extriangulated categories were introduced by Nakaoka and Palu by extracting the similarities between exact categories and triangulated categories. A notion of mutation of subcategories in an extriangulated category is defined in this paper. Let $\mathcal{A}$ be an extension closed subcategory of an extriangulated category $\mathscr{C}$. Then the additive quotient category $\mathcal{M} := \mathcal{A}/[\mathscr{X}]$ carries naturally a triangulated structure whenever $(\mathcal{A}, \mathcal{A})$ forms an $\mathscr{X}$-mutation pair. This result generalizes many results of the same type for triangulated categories. It is used to give a classification of thick triangulated subcategories of pretriangulated category $\mathscr{C}/[\mathscr{X}]$, where $\mathscr{X}$ is functorially finite in $\mathscr{C}$. When $\mathscr{C}$ has Auslander–Reiten translation τ, we prove that for a functorially finite subcategory $\mathscr{X}$ of $\mathscr{C}$ containing projectives and injectives, the quotient $\mathscr{C}/[\mathscr{X}]$ is a triangulated category if and only if $(\mathscr{C}, \mathscr{C})$ is $\mathscr{X}$-mutation, and if and only if $\tau\underline{\mathscr{X}} = \overline{\mathscr{X}}$. This generalizes a result by Jørgensen who proved the equivalence between the first and the third conditions for triangulated categories. Furthermore, we show that for such a subcategory $\mathscr{X}$ of the extriangulated category $\mathscr{C}$, $\mathscr{C}$ admits a new extriangulated structure such that $\mathscr{C}$ is a Frobenius extriangulated category. Applications to exact categories and triangulated categories are given. From the applications

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we present extriangulated categories which are neither exact categories nor triangulated categories.

1. Introduction

Exact and triangulated categories are two fundamental structures in algebra and geometry. A first looking at their construction shows that these two kinds of categories have many different points: while modules or sheaves are forming abelian categories which are exact, complexes lead to homotopy or derived categories that are triangulated. But if we look carefully at the construction inside the categories, they share many similarities: while exact categories admit short exact sequences, triangulated categories admit triangles. The similarity between short exact sequences and triangles makes that it is possible to put these two notations into a unified form. This was carried out recently by Nakaoka and Palu [20]. They introduce the new notion of extriangulated categories. The class of extriangulated categories contains exact categories and extension closed subcategories of triangulated categories as examples, it is closed under taking some quotients (see [20]).

We want to show further the similarity and difference between exact categories and triangulated categories from the point of view of construction of additive quotient triangulated categories.

Additive quotients of exact or triangulated categories have been extensively studied in representation theory. It is in particular very much of interest to know which structure of the quotient inherits from the exact or triangulated category one starts with. Well-studied examples include:

- stable categories, which are triangulated quotients of Frobenius exact categories. Happel [12] shows that if $(\mathcal{B}, \mathcal{S})$ is a Frobenius exact category, then its stable category $\mathcal{B}/[\mathcal{I}]$ carries a triangulated structure, where $\mathcal{I}$ is the full subcategory of $\mathcal{B}$ consisting of $\mathcal{S}$-injective objects. Beligiannis obtained a similar result [5, Theorem 7.2] by replacing $\mathcal{B}$ with a triangulated category $\mathscr{C}$ and replacing $\mathcal{S}$ with a proper class of triangles $\mathcal{E}$.
- certain subquotient categories of triangulated categories formed from mutations in triangulated categories. Mutation of cluster tilting objects introduced by Buan–Marsh–Reineke–Reiten–Todorov [8] and by Geiß–Leclerc–Shrör [11] is a key ingredient in attempting to categorify Fomin–Zelevinsky's cluster algebras [10] by using quiver representations. The notion of mutation pairs of subcategories in a triangulated category was defined by Iyama and Yoshino [13], which is a generalization of mutation of cluster tilting objects in cluster categories [8]. We recall the Iyama–Yoshino's definition here.