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Gorenstein liaison for toric ideals of graphs ☆

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A B S T R A C T

A central question in liaison theory asks whether every Cohen–Macaulay, graded ideal of a standard graded $\mathbb{K}$-algebra belongs to the same G-liaison class of a complete intersection. In this paper we answer this question positively for toric ideals defining edge subrings of bipartite graphs.

Introduction

Let $\mathbb{K}$ be a field and S be a standard graded $\mathbb{K}$-algebra. A central question in liaison theory asks whether every Cohen–Macaulay, graded ideal of S belongs to the same G-liaison class of a complete intersection. The question has been answered in the affirmative in several cases of interest, including for ideals of height two [5], Gorenstein ideals [21], [1], special families of monomial ideals [15], [13], [17], generically Gorenstein ideals containing a linear form [16], and several families of ideals with a determinantal or pfaffian structure [12], [6], [7], [8], [4], [9]. The argument is often inductive, meaning that an ideal

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of the family is linked to another one with smaller invariants, and the ideals with the smallest invariants are complete intersections. For example, let $m \leq n$ and consider an ideal of height $n-m+1$ generated by the maximal minors of an $m \times n$ matrix. Any such ideal is G-linked in two steps to an ideal of the same height, generated by the maximal minors of an $(m-1) \times (n-1)$ matrix, and the ideals of height $n-m+1$ generated by the entries of a $1 \times (n-m+1)$ matrix are complete intersections. In this paper, we apply a similar approach to a family of ideals associated to graphs.

There are several ways of associating a binomial ideal to a graph [20,18]. Here we consider the ideal $P(G)$ defining the edge subring $\mathbb{K}[G]$ of G, that is the $\mathbb{K}$-algebra whose generators correspond to the edges of the graph, and whose relations correspond to the even closed walks. For a survey on the importance of these rings we refer to [20, Chapters 10 and 11]. These binomial ideals are prime and Cohen–Macaulay, for all bipartite graphs. We prove that they belong to the G-biliaison class of a complete intersection. This implies in particular that they can be G-linked to a complete intersection in an even number of steps. An interesting feature of the liaison steps that we produce is that the same steps link the corresponding initial ideals, with respect to an appropriate order. In particular, the initial ideals are Cohen–Macaulay. Understanding the G-liaison pattern of the initial ideals allows us also to show that the associated simplicial complexes are vertex decomposable. For the determinantal and pfaffian ideals discussed above, the same behavior in terms of linkage of initial ideals and vertex decomposability was shown in [10].

1. Notation and preliminaries

For a positive integer n, we denote by $[n]$ the set $\{1, \ldots, n\}$. Let G be a graph with vertex set $V(G) = [n]$ and edge set $E(G) \subseteq 2^{[n]}$. We denote by q_G (or just q, if no confusion arises) the number of edges of G. The *local degree* $\rho(v)$ of v is the number of edges incident to v. A *leaf* is a vertex of local degree 1. A graph is *bipartite* if its vertex set $V(G) = V_1 \sqcup V_2$ is a disjoint union of two sets, such that every edge joins a vertex from V_1 with a vertex from V_2. It is well known that a graph is bipartite if and only if it does not contain odd cycles.

Definition 1.1. A *walk* of length m in G is an alternating sequence of vertices and edges

$$\mathbf{w} = \{v_0, e_1, v_1, \ldots, v_{m-1}, e_m, v_m\},$$

where $e_k = \{v_{k-1}, v_k\}$ for all $k = 1, \ldots, m$. A walk may also be written as a sequence of vertices with the edges omitted, or vice-versa. If $v_0 = v_m$, then $\mathbf{w}$ is a *closed walk*. A walk is called *even* (respectively *odd*) if its length is even (respectively odd). A walk is called a *path* if its vertices are distinct. A *cycle* in G is a closed walk $\{v_0, e_1, v_1, \ldots, v_m\}$ in which the vertices $v_1, \ldots, v_m$ are distinct. Denote by $\mathcal{C}(G)$ the set of even cycles of G.