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Algorithms for embedded monoids and base point free problems

Anne Fahrner [1]

Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany

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ABSTRACT

We present algorithms for basic computations with monoids in finitely generated abelian groups such as monoid membership testing and computing an element of the conductor ideal. Applying them to Mori dream spaces, we obtain algorithms to test whether a Weil divisor class of a given Mori dream space is base point free, to compute generators of the monoid of base point free Cartier divisor classes and to test whether a $\mathbb{Q}$-factorial Mori dream space with known canonical class fulfills Fujita's base point free conjecture or not.

1. Introduction

A first part of this paper concerns *embedded monoids*, that means finitely generated monoids in finitely generated abelian groups, and thereby generalizes ideas of the theory on affine semigroups [6, Chapter 2] to monoids with non-trivial torsion part. We further present algorithms for embedded monoids, among others for computing generators of

E-mail address: fahrner@math.uni-tuebingen.de.

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intersections of embedded monoids and for computing an element of the conductor ideal; see Algorithms 2.6–2.12.

In the second part of the paper, we apply these algorithms to base point free questions for Mori dream spaces. Recall that Mori dream spaces, introduced by Hu and Keel [18], are characterized via their optimal behavior with respect to the minimal model program. A particular interesting aspect of Mori dream spaces is their highly combinatorial structure [2] – in this regard they are a canonical generalization of toric varieties. Further well-known example classes are spherical varieties [5], smooth Fano varieties [3] and all Calabi–Yau varieties of dimension at most three and with polyhedral effective cone [24]. The combinatorial framework developed in [2] allows algorithmic treatment of Mori dream spaces. Applying the aforementioned algorithms to Mori dream spaces, we provide algorithms for testing whether a given Weil divisor class is base point free and for computing generators of the *base point free monoid*, i.e. the monoid of base point free Cartier divisor classes; see Algorithms 3.3 and 3.4.

These algorithms, together with the non-emptiness of the conductor ideal of the base point free monoid, play an important role in our main algorithm, Algorithm 4.5, testing Fujita's base point free conjecture [15]: this much studied conjecture claims that for a smooth projective variety with canonical class $\mathcal{K}_X$, the Weil divisor class $\mathcal{K}_X + m\mathcal{L}$ is base point free for all ample Cartier divisor classes $\mathcal{L}$ and for all $m \geq \dim(X) + 1$. So far it is known to hold for smooth projective varieties up to dimension five [26,10,20,27]. For toric varieties with arbitrary singularities, Fujino [14] presented a proof of Fujita's base point free conjecture. Despite this substantial progress, Fujita's base point free conjecture remains in general still open. With Algorithm 4.5, we provide a tool for its algorithmic testing for $\mathbb{Q}$-factorial Mori dream spaces. Since our algorithm makes use of the canonical class $\mathcal{K}_X$, it applies to Mori dream spaces with known $\mathcal{K}_X$. This case appears quite often: for instance if X is spherical or if its Cox ring is a complete intersection, see Remark 4.2 for details.

In [12], we provide an implementation of our algorithms building on the two `Maple`-based software packages `convex` [11] and `MDSpackage` [17]. Using this implementation, we prove Fujita's base point free conjecture for a six-dimensional Mori dream space in Example 4.6, and in Example 4.7, we study a locally factorial Mori dream space that does not fulfill Fujita's base point free conjecture. In addition, we study the more general question of the existence of semiample Cartier divisor classes that are not base point free. It is well-known that for Cartier divisors on complete toric varieties, semiampleness implies base point freeness. For smooth rational projective varieties with a torus action of complexity one and Picard number two, the same statement follows immediately from the classification done in [13]. In Example 3.5, we present a first example of a smooth surface of Picard number twelve admitting a semiample Cartier divisor with base points.

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